Introduction to d=2 CFT

Ref. Di Francesco, Mathieu, Senechal ; Ginsparg ; Zamolodchikov's

The "most solved" QFT ever & by far

- QFT is basic tool for quantum phenomena in space-time.
- Not so much are solved : trivial, no quantum (BPS), approx. (class. pert.)
- 2d CFT provides insights modern hydrogen atom gave to chemists

Old subject (~40 yrs but not older than QE, QFT)
getting more important for many reasons NOW

* Introduction : 10 hrs are NOT enough!
(will cover essentials only without rigor)

Why important ?

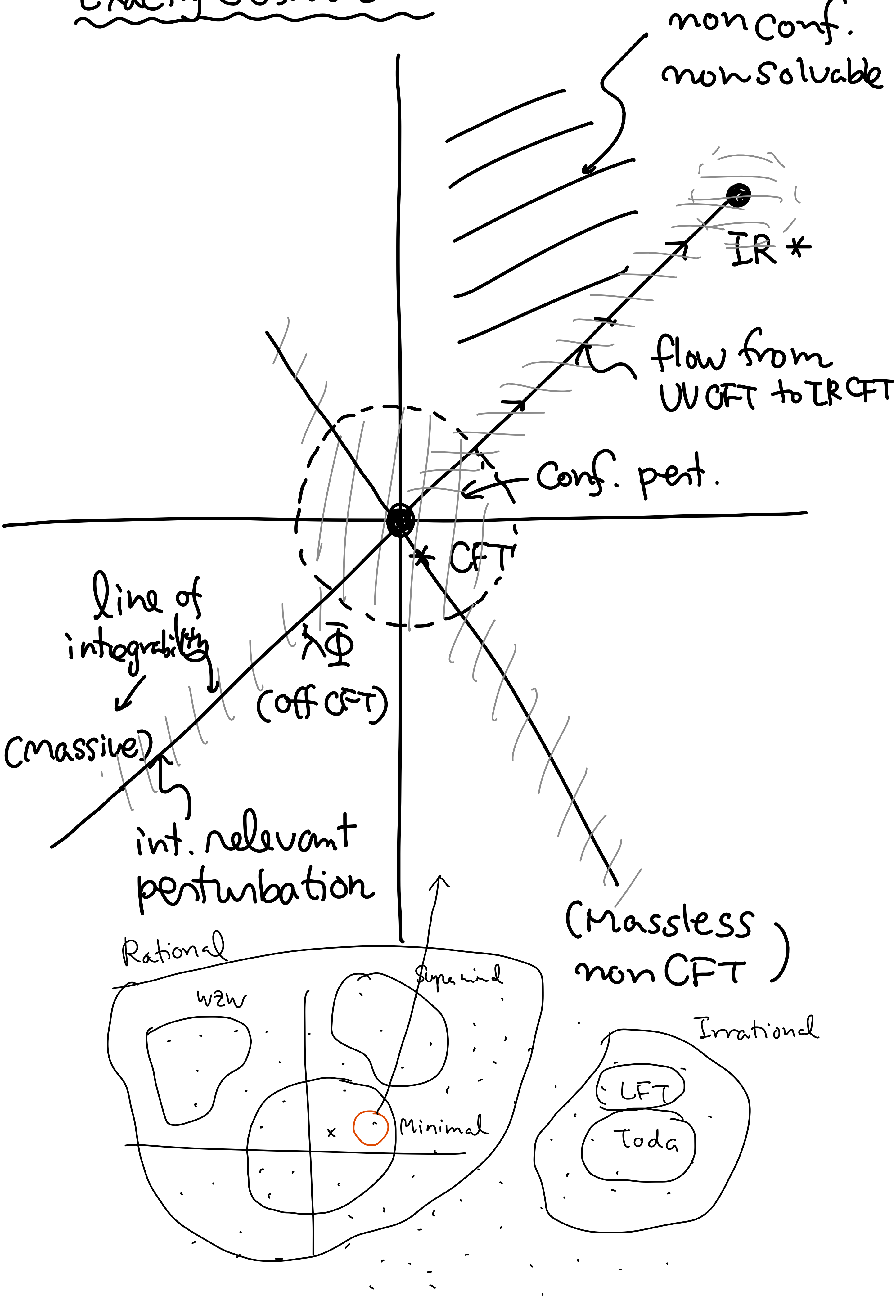
- Role model for successful application of symmetries to exact computation
- Provides paradigm for other QFTs

Why d=2 CFT ?

- String worldsheet
- 2d Stat Mech at criticality
- Experimental techniques for low dim. system
  - intrinsic 1D material, effective 1D. (Kondo, Edge state)
  - spin chains
- Duality : AGT, AdS/CFT ... / higher D CFT.

Space of 2d QFT

Exactly Solvable



Quantum Field Theory

$\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi) \longrightarrow S = \int d^d x_E \, \mathcal{L}$

physical observables : Correlation functions.

$$\langle 0 | T[\phi_1(x_1) \cdots \phi_n(x_n)] | 0 \rangle = \frac{1}{Z} \int [D\phi] \, e^{-S} \phi_1(x_1) \cdots \phi_n(x_n)$$

↑ second quantized operators

$$Z = \int [D\phi] \, e^{-S}$$

We consider transformations of $\phi(x)$ in path integ.

$$x^\mu \longrightarrow x'^\mu \quad ; \quad \phi(x) \longrightarrow \phi'(x') = F[\phi(x)]$$

$$S' = \int d^d x \, \mathcal{L}(\phi'(x), \partial_\mu \phi') = \int d^d x' \, \mathcal{L}(\phi'(x'), \partial'_\mu \phi'(x'))$$

$$= \int d^d x' \, \mathcal{L}(F(\phi), \partial'_\mu F(\phi)(x))$$

$$= \int d^d x \, \left| \frac{\partial x'}{\partial x} \right| \mathcal{L}\left(F(\phi), \left(\frac{\partial x^\nu}{\partial x'^\mu}\right) \partial_\nu F(\phi)(x)\right)$$

(Ex) ① $x' = x + a \qquad \phi'(x+a) = \phi(x) \qquad F = 1, \; \frac{\partial x'}{\partial x} = 1$

or $\phi'(x) = \phi(x-a)$

$S' = S$ invariant.

② Lorentz $\quad x'^\mu = \Lambda^\mu{}_\nu x^\nu \qquad \phi'(\Lambda x) = (L_\Lambda \phi)(x)$

$(\phi'(x) = (L_\Lambda \phi)(\Lambda^{-1} x))$

$$S' = \int d^d x \, \mathcal{L}(L_\Lambda \phi, \Lambda^{-1} \cdot \partial (L_\Lambda \phi))$$

ϕ : scalar $\quad L_\Lambda = 1 \quad \rightarrow S' = S$ if $(\Lambda^{-1} \partial)_\mu$ appears invariant

③ scale : $\quad x' = \lambda x$

$\phi'(x') = \lambda^{-\Delta} \phi(x) \equiv F[\phi(x)] \rightarrow \underline{\phi'(x) = \lambda^{-\Delta} \phi(\lambda^{-1} x)}$

$$S' = \int d^d x \, \lambda^d \mathcal{L}(\lambda^{-\Delta} \phi, \lambda^{-1-\Delta} \partial_\mu \phi)$$

Infinitessimal transf

$$X^\mu \to x'^\mu = X^\mu + \varepsilon^\mu_{(x)} ; \qquad \delta\phi(x) \equiv \phi'(x') - \phi(x) = F[\phi(x)] - \phi(x)$$

$$S' = \int d^d x \left|\frac{\partial x'}{\partial x}\right| \mathcal{L}\left(F(\phi), \left(\frac{\partial x^\nu}{\partial x'^\mu}\right)\partial_\nu F(\phi)(x)\right)$$

$$= \int d^d x \underbrace{\det\left|\delta^\mu_\rho + \partial_\rho \varepsilon^\mu\right|}_{1 + \mathrm{Tr}\, \partial_\rho \varepsilon^\mu = 1 + (\partial_\mu \varepsilon^\mu)} \underbrace{\mathcal{L}\left(\phi + \delta\phi, (\delta^\nu_\mu - \partial^\nu \varepsilon_\mu)(\partial_\nu\phi + \partial_\nu\delta\phi)\right)}$$

$$= \int d^d x \,(1 + \partial\cdot\varepsilon)\left[\mathcal{L}(\phi, \partial_\mu\phi) + \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\left(\partial_\mu(\delta\phi) - \partial^\nu\varepsilon_\mu \partial_\nu\phi\right)\right]$$

$$S' - S \equiv \delta S = \int d^d x \left[\partial_\mu \varepsilon^\mu \mathcal{L} + \underbrace{\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\delta\phi)}_{\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right)} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\varepsilon_\mu \partial_\nu\phi\right]$$

use E-L eq; $\dfrac{\partial\mathcal{L}}{\partial\phi} = \partial_\mu \dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}$

$$= \int d^d x \left\{\left[\left(\mathcal{L}\,\delta^\mu_\nu - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi\right)\partial_\mu\varepsilon^\nu\right] + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right)\right\}$$

define Energy-Momentum (or Stress-E) tensor

$$\boxed{T^\mu_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \delta^\mu_\nu\mathcal{L}}$$

$$\delta S = \int d^d x \left\{(\partial_\mu T^\mu_\nu)\cdot\varepsilon^\nu + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right)\right\}$$

partial integ.

for translation: $\varepsilon^\nu = a^\nu$ const & $\delta\phi = 0$

$$\therefore \partial_\mu T^\mu_\nu = 0 \qquad \text{E-M tensor is conserved}$$

$$\therefore = \int d^d x \left\{\partial_\mu\Big[\underbrace{T^\mu_\nu \varepsilon^\nu_{(x)} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi(x)}_{\equiv J^\mu_\varepsilon}\Big]\right\} \neq 0 \text{ in general}$$

Symmetry of correlation

$\phi(x) \to \phi'(x') \equiv F[\phi(x)]$ as $x \to x'$; $S[\phi'] = S[\phi]$

consider $\langle \phi_1(x_1') \cdots \phi_n(x_n') \rangle$

ch. of int. variable $\int d^d x [(\partial_\mu \phi)^2 - V(\phi)]$

$= \int d^d x' [(\partial'_\mu \phi')^2 - V(\phi'(x'))]$

$= S'(\phi')$ if $\phi'(x') = F(\phi)$

$= \frac{1}{Z} \int [D\phi]\, e^{-S[\phi]} \phi_1(x_1') \cdots \phi_n(x_n')$, change of name $\phi \to \phi'$

$= \frac{1}{Z} \int \underset{[D\phi]}{[D\phi']}\, e^{-S[\phi']} \phi_1'(x_1') \cdots \phi_n'(x_n') = \frac{1}{Z} \int [D\phi]\, e^{-S[\phi']} \underbrace{\phi_1'(x_1')}_{F[\phi_1](x_1)} \cdots \phi_n'(x_n')$

$= \langle F[\phi_1](x_1) \cdots F[\phi_n](x_n) \rangle$

translation : $F[\phi] = \phi \quad x' = x + a$

$\langle \phi_1(x_1 + a) \cdots \phi_n(x_n + a) \rangle = \langle \phi_1(x_1) \cdots \phi_n(x_n) \rangle$

Lorentz for scalar

$\langle \phi_1(\Lambda x_1) \cdots \phi_n(\Lambda x_n) \rangle = \langle \phi_1(x_1) \cdots \phi_n(x_n) \rangle$

scale invariance : $x' = \lambda x \quad F[\phi] = \lambda^{-\Delta} \phi$

$\langle \phi_1(\lambda x_1) \cdots \phi_n(\lambda x_n) \rangle = \lambda^{-\Delta_1 - \cdots - \Delta_n} \langle \phi(x_1) \cdots \phi_n(x_n) \rangle$

$\delta S \neq 0$

$\langle \phi_1(x_1) \cdots \phi_n(x_n) \rangle \equiv \frac{1}{Z} \int e^{-S[\phi]} \phi_1(x_1) \cdots \phi_n(x_n) [D\phi]$

$= \frac{1}{Z} \int e^{-S'[\phi']} \phi_1'(x_1) \cdots \phi_n'(x_n) [D\phi']$ [change of int. var $\phi \to \phi'$]

$S[\phi] = \int d^d x\, \mathcal{L}(\phi(x), \partial_\mu \phi(x)) = \int d^d x' \mathcal{L}(\phi(x'), \partial_\mu \phi'(x)) = S'[\phi'] = S[\phi] + \delta S$

$\phi_i'(x_i) - \phi_i(x_i) \equiv \Delta \phi_i(x_i)$

$= \frac{1}{Z} \int [D\phi]\, e^{-S - \delta S} (\phi_1(x_1) + \Delta \phi_1(x_1)) \cdots (\phi_n(x_n) + \Delta \phi_n(x_n))$

$\cancel{\langle X \rangle} = \cancel{\langle X \rangle} + \langle (-\delta S) X \rangle + \langle \delta X \rangle$

$\delta X \equiv \sum_{j=1}^{n} \phi_1 \cdots \Delta \phi_j \cdots \phi_n$

$$\boxed{\left\langle \left(\int d^d x\, \partial_\mu J^\mu_\varepsilon \right) X \right\rangle = \langle \delta X \rangle}$$

Ward Id[1].

$\to \sum_i \partial^i_\mu \langle X \rangle = 0$

(ex) $x \to x + a$; $\Delta \phi_i = -a^\mu \partial_\mu \phi_i \quad (\partial_\mu J^\mu_\varepsilon = 0)$

$$\left\langle \left(\int d^d x \, \partial_\mu J^\mu_\varepsilon \right) X \right\rangle = \langle \delta X \rangle = \sum_i \langle \phi_1(x_1) \cdots \Delta\phi_i(x_i) \cdots \rangle$$

$$= \int d^d x \sum_i \delta^{(d)}(x - x_i) \langle \phi_1 \cdots \Delta_\varepsilon \phi_i \cdots \rangle$$

$$\therefore \quad \boxed{\partial_\mu \langle J^\mu_\varepsilon (x) \overbrace{\phi_1(x_1) \cdots \phi_n(x_n)}^{X} \rangle = \sum_{i=1}^{n} \delta^{(d)}(x - x_i) \langle \phi_1(x_1) \cdots \Delta\phi_i(x_i) \cdots \rangle}$$

Ward Identity [2]

$$\Delta\phi_i = -a^\nu \frac{\partial}{\partial x_i^\nu} \phi_i$$

① $\varepsilon^\nu = a^\nu$ $\qquad J^\mu_\varepsilon = T^\mu{}_\nu \, a^\nu$

$$\boxed{\partial_\mu \langle T^\mu{}_\nu X \rangle = -\sum_i \delta^{(d)}(x - x_i) \frac{\partial}{\partial x_i^\nu} \langle X \rangle}$$

② $\varepsilon^\nu = \omega^\nu{}_\rho x^\rho$ (Lorentz transf)

$$J^\mu_\varepsilon = T^\mu{}_\nu \underbrace{\omega^\nu{}_\rho}_{\text{antisym.}} x^\rho = \frac{\omega^\nu{}_\rho}{2} \left(T^\mu{}_\nu x^\rho - T^\mu{}_\rho x^\nu \right)$$

$$\phi'(x) = \phi(\Lambda^{-1} x) = \phi(x^\nu - \omega^\nu{}_\rho x^\rho) \Rightarrow \Delta\phi = \underbrace{-\omega^\nu{}_\rho x^\rho \frac{\partial}{\partial x^\nu} \phi}_{\frac{\omega^\nu{}_\rho}{2}(x^\rho \partial_\nu - x^\nu \partial_\rho)}$$

$$\partial_\mu \langle (T^\mu{}_\nu x^\rho - T^\mu{}_\rho x^\nu) X \rangle = -\sum_i \delta^{(d)}(x - x_i) \left\{ (x^\rho \partial_\nu - x^\nu \partial_\rho)_i \langle X \rangle + i S_i^{\nu\rho} \langle X \rangle \right\}$$

if ϕ is not scalar

combine with ①

$$\langle (T^{\rho\nu} - T^{\nu\rho}) X \rangle = -i \sum_i \delta^{(d)}(x - x_i) \underbrace{S_i^{\nu\rho}}_{\hookrightarrow \, s_i \varepsilon^{\nu\rho}} \langle X \rangle$$

$$\varepsilon_{\mu\nu} \langle T^{\mu\nu} X \rangle = -i \sum_{i=1}^{n} s_i \, \delta^{(d)}(x - x_i) \langle X \rangle$$

③ $\varepsilon^\nu = \varepsilon x^\nu$

$$\phi'(x) = \lambda^{-\Delta}\phi(\lambda^{-1}x)$$
$$= (1-\Delta\varepsilon)\,\phi((1-\varepsilon)x)$$
$$= \phi \underbrace{- \Delta\varepsilon\phi - \varepsilon x\cdot\partial\phi}_{\Delta\phi}$$

$$\partial_\mu \langle T^\mu{}_\nu x^\nu X \rangle = -\sum_i \delta^{(d)}(x-x_i) \left\langle \cdots \left(\Delta_i\phi_i + x_i^\nu \frac{\partial\phi_i}{\partial x_i^\nu}\right)\cdots \right\rangle$$

$$= -\sum_i \delta^{(d)}(x-x_i)\left(\Delta_i + x_i^\nu \frac{\partial}{\partial x_i^\nu}\right)\langle X \rangle$$

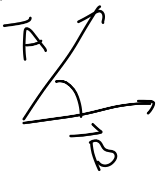
combine with ①

$$\langle T^\mu{}_\mu X \rangle = -\sum_i \delta^{(d)}(x-x_i)\,\Delta_i \langle X \rangle$$

Conformal trasf: $x_\mu \to x_\mu + \varepsilon_\mu = \tilde{x}_\mu$

$$e^{\omega(x)} \approx 1+\omega(x)$$

conformal : angle is preserved :



$$\frac{A\cdot B = g_{\mu\nu}A^\mu B^\nu}{\sqrt{(A\cdot A)(B\cdot B)}} \quad \text{if} \quad \tilde{g}_{\mu\nu} = \Omega(x)\, g_{\mu\nu} = \overbrace{\frac{\partial x^\lambda}{\partial \tilde{x}^\mu}\frac{\partial x^\rho}{\partial \tilde{x}^\nu}}^{\Omega(x)\,\delta^\lambda_\mu \delta^\rho_\nu} g_{\lambda\rho}$$

$$\frac{\partial x^\lambda}{\partial \tilde{x}^\mu} = \left(\frac{\partial \tilde{x}^\mu}{\partial x^\lambda}\right)^{-1} = (\delta_{\mu\lambda} + \partial_\lambda \varepsilon_\mu)^{-1} \cong \delta_{\mu\lambda} - \partial_\lambda \varepsilon_\mu$$

$$\therefore (1+\omega)\, g_{\mu\nu} = (\delta_{\mu\lambda} - \partial_\lambda \varepsilon_\mu)(\delta_{\nu\rho} - \partial_\rho \varepsilon_\nu)\, g_{\lambda\rho}$$

let $g_{\mu\nu} = \delta_{\mu\nu}$ (flat, Euclidean)

$$(1+\omega)\,\delta_{\mu\nu} = \delta_{\mu\nu} - \partial_\nu \varepsilon_\mu - \partial_\mu \varepsilon_\nu$$

$$-\omega\, \delta_{\mu\nu} = \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu$$

trace : $-\omega d = 2(\partial\cdot\varepsilon) \to -\omega = \frac{2}{d}(\partial\cdot\varepsilon)$

$$\therefore \quad \boxed{\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = \frac{2}{d}(\partial\cdot\varepsilon)\,\delta_{\mu\nu}}$$

∂^μ : $\quad \Box \varepsilon_\nu + \partial_\nu(\partial\cdot\varepsilon) = \frac{2}{d}\partial_\nu(\partial\cdot\varepsilon)$

$\Box \varepsilon_\nu + \frac{d-2}{d}\partial_\nu(\partial\cdot\varepsilon) = 0 \quad \to \partial^\nu$; $\Box(\partial\cdot\varepsilon) + \frac{d-2}{d}\Box(\partial\cdot\varepsilon) = 0$

$\downarrow$

$\Box(\partial\cdot\varepsilon) = 0$

∂_μ : $\quad \Box \partial_\mu \varepsilon_\nu + \frac{d-2}{d}\partial_\mu\partial_\nu(\partial\cdot\varepsilon) = 0$

$\mu \Leftrightarrow \nu \quad + \big) \; \Box \partial_\nu \varepsilon_\mu + \frac{d-2}{d}\partial_\mu\partial_\nu(\partial\cdot\varepsilon) = 0$

$$\frac{2}{d}\Box(\partial\cdot\varepsilon)\,\delta_{\mu\nu} + 2\,\frac{d-2}{d}\,\partial_\mu\partial_\nu(\partial\cdot\varepsilon) = 0$$

$$\to \left[\delta_{\mu\nu}\Box + (d-2)\,\partial_\mu\partial_\nu\right](\partial\cdot\varepsilon) = 0 \quad \longrightarrow \quad \partial_\mu\partial_\nu(\partial\cdot\varepsilon) = 0$$

if $d>2$: (1) $\mu\neq\nu \qquad \partial_\mu \varepsilon_\nu = -\partial_\nu \varepsilon_\mu$

(2) $\mu=\nu \qquad \underset{\text{(not sum)}}{\partial_\mu \varepsilon_\mu} = \frac{1}{d}(\partial\cdot\varepsilon)$ for all μ, indep of μ

$$\therefore_{\mu\neq\nu}\; \underline{\partial_\mu\partial_\nu\partial_\rho \varepsilon_\rho = 0}$$

$\Rightarrow$ ① $\varepsilon_\mu = a_\mu$ (const) [translation]

$(\partial_\mu \varepsilon_\mu = 0)$ $(\partial_\nu \varepsilon_\mu = 0)$

Conformal Poincare group

[Rotation + boost]

② $\varepsilon_\mu = \omega_{\mu\nu} x_\nu$

antisym.

$\partial_\nu \varepsilon_\mu = \omega_{\mu\nu} = -\omega_{\nu\mu}$

$\partial_\mu \varepsilon_\mu = 0$.

③ $\varepsilon_\mu = \lambda x_\mu$

$\partial_\nu \varepsilon_\mu = 0 \quad \partial_\mu \varepsilon_\mu = \lambda$

④ $\varepsilon_\mu = b_\mu x^2 + x_\mu (C\cdot x) \quad = b_\mu x^2 - 2 x_\mu (b\cdot x)$

$\partial_\mu \varepsilon_\mu = 2 b_\mu x_\mu + (C\cdot x) + C_\mu x_\mu$ ✓

$$\boxed{\tilde{x}^\mu = \frac{x^\mu + b^\mu x^2}{1 + 2b\cdot x + b^2 x^2}}$$

$= \text{indep of } \mu \quad \rightarrow \boxed{C_\mu + 2b_\mu = 0}$

$\partial_\nu \varepsilon_\mu = 2 b_\mu x_\nu + C_\nu x_\mu$

$= 2(b_\mu x_\nu - b_\nu x_\mu) = -\partial_\mu \varepsilon_\nu$ ✓

① + ② + ③ + ④

$\downarrow$

Lorentz

Poincare

Conformal group.

2d $\quad \partial_1 \epsilon_1 = \partial_2 \epsilon_2 \quad \partial_1 \epsilon_2 = -\partial_2 \epsilon_1$

$z = x_1 + i x_2 \qquad z \to z + \epsilon, \ \epsilon = \epsilon_1 + i\epsilon_2$

$\bar{z} = x_1 - i x_2 \qquad \bar{z} \to \bar{z} + \bar{\epsilon}, \ \bar{\epsilon} = \epsilon_1 - i\epsilon_2$

$x_\mu \to x_\mu + \epsilon_\mu(x)$

$$\left.\begin{aligned} &\partial_z \bar{\epsilon} = 0 \\ &\partial_{\bar{z}} \epsilon = 0 \end{aligned}\right\} \quad \partial_z = \tfrac{1}{2}(\partial_1 - i\partial_2)$$

$\therefore \epsilon = \epsilon(z), \ \bar{\epsilon} = \bar{\epsilon}(\bar{z})$

$z \to z + \epsilon(z) = w$ $\qquad$ Conformal

$\bar{z} \to \bar{z} + \bar{\epsilon}(\bar{z}) = \bar{w}$

$$ds^2 = dz\,d\bar{z} \to \underbrace{\left|\frac{\partial w}{\partial z}\right|^2}_{\Omega(z,\bar{z})} dz\,d\bar{z} = dw\,d\bar{w}$$

$\epsilon = \epsilon_n = -z^{n+1}$

$$F(z+\epsilon) = F(z) + \boxed{\epsilon\,\partial_z} F \Rightarrow \delta F = \underbrace{-z^{n+1}\partial_z}_{\substack{\shortparallel \\ l_n}} F$$

similarly $\bar{l}_m$

$$[l_m, l_n] = \left[-z^{m+1}\partial_z, -z^{n+1}\partial_z\right] = (n+1-(m+1))\, z^{n+m+1}\partial_z$$

$-(m-n)$

$$= (m-n)\, l_{n+m}$$

this is classical Virasoro. "quantum" will include "central charge"

z & $\bar{z}$ "independent" "chiral" algebra

but $\bar{z} = z^*$ impose extra constraints