References

1. T. H. R. Skyrme, *Proc. R. Soc.* **A260**, 127 (1961); E. Witten, *Nucl. Phys.* **B223**, 422, 433 (1983).
2. P. Jain, J. Schechter and R. Sorkin, *Phys. Rev.* **D39**, 998 (1989); **D41**, 3855 (1990).
3. P. Jain, *Phys. Rev.* **D41**, 3527 (1990).
4. H. Asano, H. Kanada and H. So, *Phys. Rev.* **D44**, 277 (1991).
5. M. F. Atiyah and N. S. Manton, *Phys. Lett.* **B222**, 438 (1989).
6. A. Kobayashi, H. Otsu and S. Sawada, Nagoya Univ. preprint DPNU 89-55 (1989); *Phys. Rev.* **D42**, 1868 (1990).
7. N. M. Chepilko, K. Fujii and A. P. Kobushkin, *Phys. Rev.* **D43**, 2391 (1991); Kiev preprint ITF-89-26E (1989).
8. J. A. Nedler and R. Mead, *Comp. J.* **7**, 308 (1965).

Modern Physics Letters A, Vol. 6, No. 38 (1991) 3509–3515

ONSAGER ALGEBRA AND INTEGRABLE LATTICE MODELS

CHANGRIM AHN and KAZUYASU SHIGEMOTO*
Newman Laboratory of Nuclear Studies, Cornell University, Ithaca, NY 14853, USA

Received 14 May 1991
Revised 9 September 1991

We derive many integrable lattice models from the Ising and superintegrable chiral Potts models using the Onsager algebra. For each of these models, we also construct a class of integrable models from the automorphisms of the Onsager algebra. The extension of the Onsager algebra and associated integrable lattice models are considered.

1. Ever since Onsager's exact solution of two-dimensional Ising model,[1] several theoretical frameworks have been developed to construct and solve the integrable models with infinite degrees of freedom, quantum field theories or lattice statistical models in the infinite lattice size limit. A most fruitful approach is based on Yang–Baxter equation which can guarantee the existence of commuting families of the transfer matrices. With Boltzmann weights satisfying the Yang–Baxter equation, many two-dimensional lattice models have been constructed.[2] Using additional formalisms of corner transfer matrices and inversion relations, exact results like the free energies for the underlying models have been derived.[3] Recently, there have been progresses on the integrable field theories in two dimensions by studying conformally invariant field theories. Perturbed by a special operator, a certain class of conformal field theories preserve the integrability and are shown to possess infinite number of conserved charges.[4] The exact solutions such as S-matrices of the models can be obtained by solving the Yang–Baxter equations.

In this paper, we propose another approach which can be efficient to find not only new integrable models but also their infinite conserved charges. We think this can be complementary to the well-established approaches mentioned above. Our formalism is based on the extension of Onsager's original method to solve the Ising model. Constructing infinite-dimensional algebra, referred to Onsager algebra, from the Hamiltonian of two-dimensional field theories, the infinite number of conserved charges can be easily derived. Furthermore, since the Onsager algebra is isomorphic to the direct sum of SU(2)'s, the conserved charges can be expressed in terms of generators of the SU(2)'s. Exact results can be obtained by diagonalizing the conserved charges. We derive both new and known integrable models and their conserved charges based on the Onsager algebra. We also consider the extended Onsager algebras which are related to general classical Lie algebras and construct corresponding integrable lattice models.

2. We start with the general Hamiltonian of the following form

$$-H = g_0 A_0 + g_1 A_1 \,, \tag{1}$$

where g_0 and g_1 are parameters and A_0 and A_1 are any two operators defined on the lattice with summation along a spatial dimension. Dolan and Grady proved that this Hamiltonian can generate infinite number of conserved charges if and only if the operators satisfy a condition, referred to the Dolan–Grady condition,[5]

$$\Big[A_1,[A_1,[A_1,A_0]]\Big] = 16[A_1,A_0]\,,\quad \Big[A_0,[A_0,[A_0,A_1]]\Big] = 16[A_0,A_1]\,. \tag{2}$$

The Ising and superintegrable N-state chiral Potts model (Z_N spin model)[6] have Hamiltonians of this form.

With the two operators A_0 and A_1 satisfying (2), one can generate the Onsager algebra;

$$[A_m, A_n] = 4G_{m-n}\,,\quad [G_m, A_n] = 2A_{n+m} - 2A_{n-m}\,,\quad [G_m, G_n] = 0\,. \tag{3}$$

Equation (3) contains not only the defining relations for operators A_m and G_m but also infinite set of constraints such as $[A_0, A_1] = [A_1, A_2] = [A_{-1}, A_0]$. These constraints are satisfied as far as the Dolan–Grady condition is satisfied. The infinite number of commuting charges are given by

$$-Q_m = \frac{1}{2}\{g_0(A_m + A_{-m}) + g_1(A_{m+1} + A_{-m+1})\} \tag{4}$$

with Q_0 as the Hamiltonian.

We generate new sets of infinite number of commuting charges by adding $g_l(A_{m+l} + A_{-m+l})$ to (4) as follows,

$$-Q_m = \frac{1}{2}\sum_l g_l(A_{m+l} + A_{-m+l})\,,\quad -Q_0 \equiv -H[g_i] = \sum_l g_l A_l\,, \tag{5}$$

with arbitrary integer l and parameter g_l. It is easy to check $[Q_m, Q_n] = 0$. Therefore, the Hamiltonians $H[g_i]$ define a family of new integrable models and Q_m's are the associated infinite number of conserved charges.

The operators A_m and G_m can be expressed by[1,7]

$$A_m = 2\sum_k (e^{-im\theta_k}E_k^+ + e^{im\theta_k}E_k^-)\,,\quad G_m = 2\sum_k (e^{-im\theta_k} - e^{im\theta_k})H_k\,, \tag{6}$$

in terms of SU(2) generators $\{E_k^\pm, H_k\}$ satisfying $[E_k^+, E_l^-] = 2\delta_{k,l}H_k$, $[E_k^\pm, H_l] = \mp\delta_{k,l}E_k^\pm$ up to additive constants. This means that the Onsager algebra is a Lie algebra which is isomorphic to the direct sum of copies of SU(2)'s defined on each site k. The explicit form of θ_k depends on the underlying model. For the Ising model, $\theta_k = 2\pi k/M$ with lattice size M and (6) corresponds to the Fourier transformed expression in the momentum space. By diagonalizing $E_k^\pm$, we can solve the integrable models in (5) exactly in terms of the eigenvalues of the conserved charges;

$$-Q_m[g_i] = 4\sum_k m_k \cos m\theta_k \sqrt{\sum_l g_l^2 + 2\sum_{l>n} g_l g_n \cos(l-n)\theta_k}\,, \tag{7}$$

up to additive constants. m_k can be any half integer of $-s_k, -s_k+1, \ldots, s_k$ where s_k is the spin of SU(2) representation.

For each integrable model in (5), we can construct a class of new integrable models due to the automorphism group $\mathcal{A}$ of the Onsager algebra (3). The automorphism group $\mathcal{A}$ is generated by T and R_φ

$$T:\begin{cases} A_m \to A_{m+1} \\ G_m \to G_m \end{cases} \quad\text{and}\quad R_\varphi:\begin{cases} A_m^+ \to A_m^+ \\ A_m^- \to \cos\varphi A_m^- + i\sin\varphi G_m \\ G_m \to \cos\varphi G_m + i\sin\varphi A_m^- \end{cases} \tag{8}$$

with arbitrary angle φ and $A_m^\pm \equiv \frac{1}{2}(A_m \pm A_{-m})$. It is easy to see that (3) is invariant under T and R_φ. The generators T and R_φ are not commuting. Therefore, any element of the group $\mathcal{A}$ can be written as a polynomial of T and R_φ; $T^{n_1}R_{\varphi_1}T^{n_2}R_{\varphi_2}\ldots T^{n_N}R_{\varphi_N}$. Under an element $a \in \mathcal{A}$, the operators A_m and G_m are transformed to $\overline{A}_m^a$ and $\overline{G}_m^a$. In general, $\overline{A}_m^a$ and $\overline{G}_m^a$ are complicated linear combinations of A_l's and G_l's and satisfy the same Onsager algebra (3).

Replacing A_m's with $\overline{A}_m^a$'s in (5), we obtain new sets of conserved charges $\overline{Q}_m^a$ and related integrable models. It is remarkable that the eigenvalues of these charges are the same as (7) because one can see that the eigenvalues are invariant under both T and R_φ. Therefore, the free energies are same for all the models generated by the automorphisms. We call this class of integrable models with the same eigenvalues of the conserved charges an equivalence class. The automorphisms act as hidden symmetries which connect all the models in the equivalence class. Each Hamiltonian $H[g_i]$ in the family of (5) generates one equivalence class of integrable models.

We emphasize that the Q_m's in Eq. (5) and $\overline{A}_m$'s above cannot be derived by manipulating the original model. Therefore, the models obtained from the above two separate generalizations describe quite independent models.

3. We apply our general formalism to the Ising and superintegrable chiral Potts models. Let's consider first the Ising model. The Ising model on two-dimensional lattice can be treated as an Euclidean field theory with the Hamiltonian[1]

$$-H = g_0 A_0 + g_1 A_1 = g_0 \sum_j \sigma_j^x + g_1 \sum_j \sigma_j^z \sigma_{j+1}^z\,. \tag{9}$$

With the operators A_m's expressed in terms of the σ-matrices, the Hamiltonians in (5) are given by

$$-H[g_i] = \sum_j \left[g_0\sigma_j^x + g_1\sigma_j^z\sigma_{j+1}^z + \sum_{m=2} (-1)^{m+1} g_m \sigma_j^z \sigma_{j+1}^x \cdots \sigma_{j+m-1}^x \sigma_{j+m}^z \right. \\ \left. + \sum_{m=1} (-1)^{m+1} g_{-m} \sigma_j^y \sigma_{j+1}^x \cdots \sigma_{j+m-1}^x \sigma_{j+m}^y \right] . \qquad (10)$$

Γhe infinite number of conserved charges can be written down similarly. These nodels in the Ising family have been known as the generalized XY models. The :igenvalues of the Hamiltonians, given in (7), agree with the known results.[8] The ntegrability of this generalized XY models was not established before. Our explicit :onstruction of an infinite number of conserved charges in (5) shows that all the nodels in the Ising family are indeed integrable. The eigenvalues of the conserved :harges are given, (7), with appropriate representations of the SU(2)'s. The free :nergies of the models in the Ising family can be derived in the same way as Onsager lid using spin-1/2 representation.[1] One of the simplest model in the Ising family is he XY model with the Hamiltonian

$$-H[g_0, g_1, g_{-1}] = g_0 \sum_j \sigma_j^x + g_1 \sum_j \sigma_j^z \sigma_{j+1}^z + g_{-1} \sum_j \sigma_j^y \sigma_{j+1}^y . \qquad (11)$$

'or this model, the conserved charges were derived using an independent method[9] nd are consistent with our results in (5). All other models in (10) contain the on-nearest neighbor interactions. Interesting quantities like the free energies can e computed exactly using the eigenvalues of the conserved charges (7).

As stressed before, the automorphism group $\mathcal{A}$ can generate an equivalence lass of integrable models for each model in the Ising family (10). Although all the ıodels in an equivalence class have the same eigenvalues, hence the same physical ɔntents, the explicit Hamiltonians in terms of the σ-matrices are different. This ıeans that there exist unknown connections between the models in the equivalence lass which appear as the automorphisms in the abstract Onsager algebra. For xample, consider the Ising model $-H[g_0, g_1]$. Any element of the automorphism roup can generate new Hamiltonian. Since the models generated by T and R_φ ın be identified with the original model by redefining the σ^y and σ^z in the y–z lane, the simplest non-trivial model is generated by the element TR_φ with the amiltonian

$$-H' = \sum_j g_0 \sigma_j^z \sigma_{j+1}^z + \frac{g_1}{2}[c_- \sigma_j^x - c_+ \sigma_j^z \sigma_{j+1}^x \sigma_{j+2}^z - \sin\varphi(\sigma_j^y \sigma_{j+1}^z + \sigma_j^z \sigma_{j+1}^y)] , \qquad (12)$$

ith $c_\pm \equiv 1 \pm \cos\varphi$. Since the Hamiltonian (12) contains the next-nearest neighbor teractions, this model is apparently different from the Ising model. There are no

well-defined transformations for the σ-matrices which can connect the two models. Only the automorphism of the Onsager algebra connects the two.

Next, we apply our formalism to the superintegrable N-state chiral Potts model. The Hamiltonian is given by

$$-H = g_0 A_0 + g_1 A_1 \\ = g_0 \frac{4}{N} \sum_j M_j + g_1 \frac{2}{N} \sum_j \sum_{n=1}^{N-1} \left(1 - i \cot\frac{\pi n}{N}\right) (\Gamma_j \Gamma_{j+1}^\dagger)^n ,$$

$$M_j = \begin{pmatrix} \frac{N-1}{2} & & & \\ & \frac{N-3}{2} & & \\ & & \ddots & \\ & & & -\frac{N-1}{2} \end{pmatrix}_j , \quad \Gamma_j = \begin{pmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & & \ddots & \\ & & & 1 & 0 \end{pmatrix}_j . \qquad (13)$$

This Hamiltonian was first derived and shown to be integrable using the Dolan–Grady condition.[6] This model has been studied also using the Boltzmann weights which satisfy the Yang–Baxter equation and the same Hamiltonian has been derived from the transfer matrix.[10] According to the general formalism in (5), we can generate a family of new integrable models by adding the terms $g_l A_l$ to the Hamiltonian. The conserved charges and their eigenvalues of these models are given by the general formulae (5)–(7) with the parameter[11] θ_k

$$\tan\frac{\theta_k}{2} = \left[\frac{\sin\varphi_k}{\sin(\varphi_k + \pi/N)}\right]^{N/2} , \quad \text{with} \quad \varphi_k = \frac{2\pi(1 - N^{-1})k}{M} . \qquad (14)$$

Again, each model obtained in this way can generate an equivalence class of integrable models due to the automorphisms of the Onsager algebra which have the same eigenvalues of the conserved charges.

For an explicit example, we consider a model with Hamiltonian $-H = g_0 A_0 + g_1 A_1 + g_{-1} A_{-1}$ where A_{-1} is given by

$$A_{-1} = -\frac{2}{N} \sum_j \sum_{n=1}^{N-1} \left(1 - i \cot\frac{\pi n}{N}\right) \begin{bmatrix} & \mathbf{1}_n \\ -\mathbf{1}_{N-n} & \end{bmatrix}_j \begin{bmatrix} & \mathbf{1}_{N-n} \\ -\mathbf{1}_n & \end{bmatrix}_{j+1} \qquad (15)$$

with $n \times n$ unit matrix $\mathbf{1}_n$. The eigenvalues of $-Q_m$'s are

$$-Q_m = 4 \sum_k m_k \cos m\theta_k \\ \times [g_0^2 + (g_1 - g_{-1})^2 + 2g_0(g_1 + g_{-1})\cos\theta_k + 4g_1 g_{-1}\cos^2\theta_k]^{1/2} , \qquad (16)$$

up to an additive constant. m_k takes value of $-s_k, -s_k + 1, \ldots, s_k$ and the θ_k is given by (14). For the free energies, the spin-1/2 representation is required for the perintegrable chiral Potts family.[7] This new model in the superintegrable N-state iral Potts family includes the XY model (11) as a special case $N = 2$.

As we have seen, the models in the Ising and chiral Potts families and in their sociated equivalence classes are exactly solvable because the Onsager algebras isomorphic to direct sum of copies of SU(2) Lie algebras. In terms of SU(2) nerators, the conserved charges can be easily diagonalized. We generalize the isager algebra to the extended Onsager algebras which are isomorphic to the neral classical Lie algebras. Our approach is different from the previous one in it we start with the extended Onsager algebras rather than Hamiltonians. After nstructing an infinite number of mutually commuting charges $-Q_m$, the integrable odels are obtained with the Hamiltonians Q_0.

Let's start with the operators which generate the extended Onsager algebra morphic to the general Lie algebra;

$$A_m^\alpha = 2\sum_j (e^{-im\theta_j} E_j^{+\alpha} + e^{im\theta_j} E_j^{-\alpha}), \quad G_m^\alpha = 2\sum_j (e^{-im\theta_j} - e^{im\theta_j}) H_j^\alpha, \quad (17)$$

h the generators of SU(2) subalgebras $\{E_k^{\pm\alpha}, H_k^\alpha\}$ of the classical Lie algebra in Chevalley bases. The corresponding extended Onsager algebra is

$$\begin{aligned} &[A_m^{\alpha_i}, A_n^{\alpha_j}] = 4\delta_{ij} G_{m-n}^{\alpha_i} + 2(1-\delta_{ij})\left(N_{\alpha_i,\alpha_j} A_{m+n}^{\alpha_i+\alpha_j} + N_{\alpha_i,-\alpha_j} A_{m-n}^{\alpha_i-\alpha_j}\right) \\ &[G_m^{\alpha_i}, A_n^{\alpha_j}] = 2C_{ij}(A_{n+m}^{\alpha_j} - A_{n-m}^{\alpha_j}), \quad [G_m^{\alpha_i}, G_n^{\alpha_j}] = 0, \end{aligned} \quad (18)$$

using the standard notations from $[E^{\alpha_i}, E^{\alpha_j}] = N_{\alpha_i,\alpha_j} E^{\alpha_i+\alpha_j}$ if $\alpha_i + \alpha_j$ is a t and $[H^{\alpha_i}, E^{\pm\alpha_j}] = \pm C_{ij} E^{\pm\alpha_j}$. From (17), the infinite number of commuting rges are

$$-Q_m = \frac{1}{2}\sum_\alpha \sum_l g_l^\alpha (A_{m+l}^\alpha + A_{-m+l}^\alpha). \quad (19)$$

To give a realization of the abstract algebra, we consider the generalization of the g model with $\theta_k = 2\pi k/M$ by introducing $2N$ fermions $(\psi_k)^a$ $((\psi_k^\dagger)^a = (\psi_{-k})^a)$. erms of these fermions, $A_m^{\alpha_i}$'s are given by

$$A_m^{\alpha_i} = 2\sum_{k>0} \left[e^{-im\theta_k} (\psi_k^\dagger)^a (E^{+\alpha_i})_{ab} (\psi_k)^b + e^{im\theta_k} (\psi_k^\dagger)^a (E^{-\alpha_i})_{ab} (\psi_k)^b\right]. \quad (20)$$

erms of the fermions p_j^a's and q_j^a's on the coordinate lattice obtained from)^a's using the Fourier transformation, we can express the operators $A_m^{\alpha_i}$'s and conserved charges $-Q_m$ by the following form,

$$-Q_m \sim \sum_j \sum_l C_{ab}^l p_j^a q_{j+m+l}^b + \cdots, \quad (21)$$

re C_{ab}^l's depend on the explicit representation for the generators. Since the ices C^l's in the flavor space do not commute with one another, the models in

(21) are different from the direct sum of the Ising models. To construct the models with the extended Onsager algebra on the lattice, we generalize the Jordan–Wigner transformation defined for the pair of fermions to the general set of fermions as follows,

$$\begin{aligned} &p_j^a = \Sigma_1^x \Sigma_2^x \cdots \Sigma_{j-1}^x \Sigma_j^{az}, \quad iq_j^a = \Sigma_1^x \Sigma_2^x \cdots \Sigma_{j-1}^x \Sigma_j^{ay} \\ &\Sigma_j^x \equiv (\sigma_j^x)^{\otimes N}, \quad \Sigma_j^{ay,z} \equiv (\sigma_j^x)^{\otimes(a-1)} \otimes \sigma_j^{y,z} \otimes \mathbf{1}^{\otimes(N-a)}. \end{aligned} \quad (22)$$

The commutation relations of the fermions on the lattice are given by

$$\{p_j^a, p_l^b\} = 2\delta^{ab}\delta_{jl}, \quad \{q_j^a, q_l^b\} = -2\delta^{ab}\delta_{jl}, \quad \{p_j^a, q_l^b\} = 0. \quad (23)$$

Using these representations of the fermions, we can express the conserved charges including the Hamiltonians in terms of the σ-matrices defined on the lattice. These models are direct generalization of the models derived in (10). In particular, we obtain the Ising family for $N = 1$.

Acknowledgments

One of us (K. S.) thanks T. Kinoshita and T.-M. Yan for their kind hospitality at Cornell University. This work was supported in part by the National Science Foundation.

References

1. L. Onsager, *Phys. Rev.* **65**, 117 (1944).
2. R. J. Baxter, *Adv. Stud. Pure Math.* **19**, 95 (1989) and references therein.
3. R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, 1982).
4. A. B. Zamolodchikov, *Int. J. Mod. Phys.* **A4**, 4235 (1989).
5. L. Dolan and M. Grady, *Phys. Rev.* **D25**, 1587 (1982).
6. G. von Gehlen and V. Rittenberg, *Nucl. Phys.* **B257** [FS14], 351 (1985).
7. B. Davis, *J. Phys.* **A23**, 2245 (1990).
8. M. Suzuki, *Phys. Lett.* **A34**, 338 (1971).
9. E. Barouch and B. Fuchssteiner, *Stud. Appl. Math.* **73**, 221 (1985); H. Araki, *Commun. Math. Phys.* **132**, 155 (1990).
10. R. J. Baxter, J. H. H. Perk and H. Au-Yang, *Phys. Lett.* **A128**, 138 (1988); G. Albertini, B. M. McCoy, J. H. H. Perk and S. Tang, *Nucl. Phys.* **B314**, 741 (1989); G. Albertini, B. M. McCoy and J. H. H. Perk, *Phys. Lett.* **A139**, 204 (1989).
11. R. J. Baxter, *Phys. Lett.* **A133**, 185 (1988); *J. Stat. Phys.* **52**, 639 (1988); **57**, 1 (1989).