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Thermodynamic Bethe ansatz equations for perturbed minimal conformal field theories

Changrim Ahn [1,2] and Soonkeon Nam [3]
Department of Physics, Center for Theoretical Physics, Seoul National University, Seoul 151, South Korea

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We study S-matrices of kinks and breathers of the restricted sine-Gordon theory using the thermodynamic Bethe ansatz method. We found that the underlying conformal field theories at the massless limit are minimal $\mathcal{M}_{p/q}$ theories, including the non-unitary as well as the unitary ones. This result confirms the identification of the restricted sine-Gordon theory with integrable perturbation of minimal $\mathcal{M}_{p/q}$ theories.

1. A large class of 2D quantum field theories (QFTs) has been associated with conformal field theories (CFTs) [1]. The CFTs correspond to the fixed points of the renormalization group of 2D QFTs. Hence, these QFTs can be thought of as CFTs perturbed by the appropriate relevant operators. Therefore, these 2D QFTs can be completely defined by the "CFT data": the underlying CFTs and their perturbing relevant operators. Of particular interest are the integrable QFTs whose S-matrices can be solved exactly.

Two important questions can be raised. One is what kind of CFTs and relevant operators can generate integrable QFTs and the other is whether we can identify the CFT data if we know the exact S-matrices of a given integrable QFT. The original approach to the first question by Zamolodchikov was to construct the conserved charges explicitly using conformal perturbation theories [2]. While this approach has been successful for a certain class of CFTs and corresponding perturbations, it is difficult to see the general relationship between CFTs and integrable QFTs. Another approach is to start with a known integrable QFT like the sine-Gordon (SG) theory. Using the quantum group structure of these theories, one can truncate the multi-particle Hilbert space while keeping the integrability of the theory. The resulting restricted integrable QFTs have been related with a certain perturbation of a wide class of CFTs [3–9]. The corresponding S-matrices are expressed in RSOS form which was introduced in 2D lattice models to construct RSOS models from vertex models [10]. In this paper, we are particularly interested in the restricted sine-Gordon (RSG) theory. It has been claimed that this theory is minimal CFT [1] perturbed by a $\Phi_{1,3}$ operator [3–7].

The thermodynamic Bethe ansatz (TBA) approach has been very successful as an answer to the second question for a restricted class of factorizable S-matrices. The TBA method consists in computing the free energy of many-particle systems in the thermodynamic limit. By computing the free energy at a very high temperature limit, one can derive the CFT data, i.e. the central charge of the underlying CFT and the conformal dimension of the perturbing operator. This approach was limited to diagonal S-matrices [11–15]. For non-diagonal S-matrix theories, Zamolodchikov partly derived the TBA equations for pure RSOS theory [16]. He conjectured TBA equations based on S-matrices of the RSG$[p/(p+1)]$ theory that correctly reproduce the CFT data of unitary CFT $\mathcal{M}_{p/(p+1)}$. The next obvious question to ask is whether the TBA method can cover the whole RSG$[p/$

[1] Permanent address: Newman Laboratory of Nuclear Studies, Cornell University, Ithaca, NY 14853, USA.
[2] E-mail address: ahn@cornella.bitnet.
[3] E-mail address: snu00049@krsnuccl.bitnet.

q] theory, which is claimed to be the integrable perturbation of minimal CFT $\mathscr{M}_{p/q}$ [7]. In this letter we address this question and show that we can derive TBA equations that reproduce the CFT data correctly.

2. The minimal CFT perturbed by the least relevant operator is claimed to be the RSG theory, obtained by truncating the SG multi-soliton Hilbert space. The action of the SG theory is

$$\mathscr{S}_{\mathrm{SG}}=\frac{1}{\beta^2}\int \mathrm{d}^2x\left(\tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi+m^2\cos\phi\right) . \tag{1}$$

The particle spectrum of the SG theory includes soliton (A), anti-soliton ($\bar{A}$) and breathers (B_n) which are bound states of A and $\bar{A}$ with $n=1, 2, ...\leqslant 8\pi/\gamma$, where $\gamma=(\beta^2/8\pi)/(1-\beta^2/8\pi)$. Due to the integrability of the SG theory, the multi-particle S-matrices can be factorized into products of two-particle elastic S-matrices between A, $\bar{A}$, B_n. These two-particle S-matrices have been derived by solving the factorization (or Yang–Baxter) equations [17].

As is well known [3], the solution of the Yang–Baxter equations can be expressed in terms of the $\mathscr{R}$-matrix of $\mathrm{SL}_q(2)$ for the SG theory. The quantum group $\mathrm{SL}_q(2)$ denotes the group SL(2) deformed by the parameter q. The deformation parameter is

$$q=-\exp(-\mathrm{i}8\pi^2/\gamma)=-\exp(-\mathrm{i}\pi/P), \quad \text{with } P\equiv\gamma/8\pi . \tag{2}$$

Therefore, $(A, \bar{A})$ and the B_n form a spin-$\frac{1}{2}$ and singlet representations of $\mathrm{SL}_q(2)$, respectively. This means that one can decompose the multi-soliton and anti-soliton Hilbert space of the SG theory into subspaces represented by higher spins:

$$\mathscr{H}=|\tfrac{1}{2}, m_1\rangle\otimes|\tfrac{1}{2}, m_2\rangle\otimes ... \otimes|\tfrac{1}{2}, m_N\rangle=\sum_J |J, M\rangle . \tag{3}$$

In the $|J, M\rangle$ basis the soliton and anti-soliton are replaced by kinks K_{ab} with $|a-b|=\frac{1}{2}$. The decomposed Hilbert space in eq. (3) can be truncated while preserving the integrability if q is a root of unity. The resulting Hilbert space which consists of subspaces with spin J satisfying $0\leqslant J\leqslant J_{\max}$ defines the Hilbert space of the RSG theory. From eq. (2), if P is an irrational number, the RSG theory is equivalent to the SG theory because q is not a root of unity [5]. If P is a rational number ($P=p/(q-p)$ with two coprime integers p, q ($q>p$)), $q^p=\pm 1$ and $J_{\max}=\frac{1}{2}p-1$. The particle spectrum of the RSG theory is

$$\text{Kinks: } K_{ab},\quad |a-b|=\tfrac{1}{2},\quad a, b=0, \tfrac{1}{2}, ..., \tfrac{1}{2}p-1 ,\qquad \text{Breathers: } B_n,\quad n=1, 2, ..., \leqslant q/p-1 . \tag{4}$$

We refer to this theory as RSG[p/q]. This theory is claimed to be the $\Phi_{1,3}$ perturbation of the minimal $\mathscr{M}_{p/q}$ CFT.

The kink–kink S-matrices are expressed in terms of an RSOS form [10,7,6]. The S-matrices of the kink–breather and breather–breather are the same as those of the SG soliton–breather and breather–breather [17] because the breathers are singlets of $\mathrm{SL}_q(2)$. The complete S-matrices of the RSG theory, as functions of rapidity θ, are summarized as follows:

Kink–Kink: $K_{da}+K_{ab}\to K_{dc}+K_{cb}$,

$$S_{dc}^{ab}(\theta)=\sigma(\theta)\left(\frac{[2a+1][2c+1]}{[2d+1][2b+1]}\right)^{-\theta/2\pi\mathrm{i}}\left[\delta_{db}\sinh\left(\frac{\theta}{P}\right)\left(\frac{[2a+1][2c+1]}{[2d+1][2b+1]}\right)^{1/2}+\delta_{ac}\sinh\left(\frac{\mathrm{i}\pi-\theta}{P}\right)\right], \tag{5a}$$

Kink–Breather: $K_{ab}+B_n\to B_n+K_{ab}$,

$$S^{(n)}(\theta)=\frac{\sinh\theta+\mathrm{i}\cos[\frac{1}{2}(n\pi P)]}{\sinh\theta-\mathrm{i}\cos[\frac{1}{2}(n\pi P)]}\prod_{l=1}^{n-1}\frac{\sin^2[\frac{1}{4}(n-2l)\pi P-\frac{1}{4}\pi+\frac{1}{2}\mathrm{i}\theta]}{\sin^2[\frac{1}{4}(n-2l)\pi P-\frac{1}{4}\pi-\frac{1}{2}\mathrm{i}\theta]}, \tag{5b}$$

Breather–Breather: $B_n+B_m \to B_m+B_n$,

$$S^{(n,m)}(\theta)=\frac{\sinh\theta+\mathrm{i}\sin[\frac{1}{2}(n+m)\pi P]}{\sinh\theta-\mathrm{i}\sin[\frac{1}{2}(n+m)\pi P]}\frac{\sinh\theta+\mathrm{i}\sin[\frac{1}{2}(n-m)\pi P]}{\sinh\theta-\mathrm{i}\sin[\frac{1}{2}(n-m)\pi P]}$$
$$\times\prod_{l=1}^{\min(m,n)-1}\frac{\sin^2[\frac{1}{4}(m-n-2l)\pi P+\frac{1}{2}\mathrm{i}\theta]\cos^2[\frac{1}{4}(m+n-2l)\pi P+\frac{1}{2}\mathrm{i}\theta]}{\sin^2[\frac{1}{4}(m-n-2l)\pi P-\frac{1}{2}\mathrm{i}\theta]\cos^2[\frac{1}{4}(m+n-2l)\pi P-\frac{1}{2}\mathrm{i}\theta]}, \tag{5c}$$

where the q-number $[n]$ is given by $[n]=(q^n-q^{-n})/(q-q^{-1})$ and the prefactor $\sigma(\theta)$ satisfies from unitary and crossing symmetry

$$\sigma(\theta)=\sigma(\mathrm{i}\pi-\theta),\quad \sigma(\theta)\sigma(-\theta)=(2\cos 2\pi/P-2\cosh 2\theta/P)^{-1}. \tag{6}$$

The RSOS kink–kink S-matrices satisfy unitarity only when the parameter P has the following values [7]:

$$P=\frac{N}{Nl+1}\quad\text{for } N=2,3,\ldots,\quad\text{or } P=\frac{3}{3l+2}\quad\text{where } l=0,1,\ldots. \tag{7}$$

Therefore, we concentrate only on these values of P. From eq. (4), the number of kinks is $N-2$ and that of breathers is l. In particular, if $l=0$, only kinks are left in the spectrum and the S-matrices in eq. (5) reduce to the pure RSOS S-matrices derived in ref. [6]. On the other hand, if $N=2$, there is no kink in the spectrum but breathers are. The corresponding S-matrices are exactly the same as those derived from the factorization equation along with the bootstrap ansatz [18].

This RSG theory is claimed to be the following perturbation of BPZ minimal models [1]:

$$\mathscr{S}_{p/q}=\mathscr{M}_{p/q}+\int \mathrm{d}z^2\,\Phi_{1,3}(z,\bar{z}). \tag{8}$$

The central charge and the lowest dimension of the primary fields are

$$c=1-\frac{6(p-q)^2}{pq},\quad \Delta_{\min}=\frac{1-(q-p)^2}{4pq}. \tag{9}$$

This identification of RSG$[p/q]$ theory with $\mathscr{S}_{p/q}$ is justified by considering the ultraviolet limit of the Green functions of the RSG theory [7]. For the case of no breathers ($l=0$), the RSG$[p/(p+1)]$ theory has been claimed to be a perturbed unitary CFT by comparing the topological charges of the primary fields with the allowed spins of the RSOS theory [3,6].

3. To formulate the thermodynamic Bethe ansatz (TBA) for the RSG$[p/q]$ theory, we consider a state vector of N kinks of mass m (N even for simplicity) and N_b breathers $\{B_n\}$ of mass m_n ($n=1,\ldots,l$) in a periodic box of length L:

$$|\Psi^{n_1,n_2,\ldots}_{a_1,a_2,\ldots}(\{\beta_i\},\{\theta_j\})\rangle\equiv|K_{a_1a_2}(\beta_1)K_{a_2a_3}(\beta_2)\ldots B_{n_1}(\theta_1)B_{n_2}(\theta_2)\ldots\rangle. \tag{10}$$

When the exchange two adjacent particles, the state vector is multiplied by the corresponding S-matrix. For $L\gg R_c$ (R_c is the Compton wavelength of the particles), if we taken one particle and exchange with all the other particles, we obtain the periodicity condition of the state vector:

$$\exp(\mathrm{i}mL\sinh\beta_k)\sum_{a'_1,\ldots}S^{a'_1,a'_2,\ldots}_{a_1,a_2,\ldots}(\beta_1,\beta_2,\ldots)\prod_{j=1}^{N_b}S^{(n_j)}(\theta_j-\beta_k)|\Psi^{n_1,n_2,\ldots}_{a'_1,a'_2,\ldots}(\{\beta_i\},\{\theta_j\})\rangle=|\Psi^{n_1,n_2,\ldots}_{a_1,a_2,\ldots}(\{\beta_i\},\{\theta_j\})\rangle,$$

$$\exp(\mathrm{i}m_{n_k}L\sinh\theta_k)\prod_{i,i\neq k}S^{(n_i,n_k)}(\theta_k-\theta_i)\prod_{j=1}^{N}S^{(n_k)}(\theta_k-\beta_j)|\Psi^{n_1,n_2,\ldots}_{a_1,a_2,\ldots}(\{\beta_i\},\{\theta_j\})\rangle=|\Psi^{n_1,n_2,\ldots}_{a_1,a_2,\ldots}(\{\beta_i\},\{\theta_j\})\rangle. \tag{11}$$

The first equation in eq. (11) is expressed in terms of elements of the RSOS transfer matrix and we need to diagonalize it. If we call the eigenvalue of the RSOS transfer matrix $\Lambda(\beta_k|\beta_1,...,\beta_N)$, eq. (11) reduces to

$$\exp(\mathrm{i}mL\sinh\beta_k)\Lambda(\beta_k|\beta_1,...,\beta_N)\prod_{j=1}^{N_b}S^{(n_j)}(\theta_j-\beta_k)=1\,,$$

$$\exp(\mathrm{i}m_{n_k}L\sinh\theta_k)\prod_{i,i\neq k}S^{(n_i,n_k)}(\theta_k-\theta_i)\prod_{j=1}^{N}S^{(n_k)}(\theta_k-\beta_j)=1\,. \tag{12}$$

The explicit expression of the eigenvalues of the RSOS transfer matrix is given in refs. [19,16]. In eq. (12), the S-matrices given in eq. (5) are diagonal; it is straightforward to write down the TBA equations except for the RSOS sector. For the RSOS sector, Zamolodchikov analyzed $(\mathrm{RSOS})_3$ (two kinks, no breather) and conjectured the TBA equation for a general number of kinks [16]. What we notice here is that if the number of the breathers is even, the deformation parameter q is independent of l as one can see in eq. (2). This simplifies the derivation of the TBA equations for two kinks and an even number of breathers.

Since $P=4/M$ ($M=4l+1$ with l even) in eq. (2), $q=-\exp(-\frac{1}{4}\mathrm{i}\pi)$, the RSOS transfer matrix has the following simple form:

$$\Lambda(\beta_k|\{\beta_i\})=\text{const.}\cdot\exp\left(-\mathrm{i}\frac{\ln 2}{2\pi}\sum_{i=1}^{N/2}(\beta_{2i-1}-\beta_{2i})\right)\prod_{i=1}^{N}\sigma(\beta_k-\beta_i)\prod_{i=1}^{N/2}\sinh[\tfrac{1}{2}M(\beta_k-x_i)]\,, \tag{13}$$

where the positions of the zeroes of $\Lambda(\beta_k)$, the x_i, are determined by the following constraint equation:

$$\prod_{k=1}^{N}\frac{\sinh[\frac{1}{2}M(y_i-\beta_k)+\frac{1}{4}\mathrm{i}\pi]}{\sinh[\frac{1}{2}M(y_i-\beta_k)-\frac{1}{4}\mathrm{i}\pi]}=\pm 1\,, \tag{14}$$

in terms of real numbers $y_i\equiv x_i+\frac{1}{2}\mathrm{i}\pi$ [16,19].

In the thermodynamic limit ($N, L\to\infty$), we introduce the rapidity densities of the kinks and breathers $\rho(\beta)$, $\rho^{(n)}(\theta)$. For each of these we introduce the actual densities $\rho_1(\beta)$, $\rho_1^{(n)}(\theta)$. Furthermore we have to introduce also the densities of the eigenvalue zeroes $P_\pm(y)$. The joint density $P(y)=P_+(y)+P_-(y)$ is the density of the solutions to eq. (14).

From eqs. (12), (14), we derive the relation between these densities,

$$2\pi\rho(\beta)=mL\cosh\beta+[\phi_\sigma*\rho_1](\beta)+\tfrac{1}{2}M[\phi_l*(P_+-P_-)](\beta)+\sum_{n=1}^{l}[\Phi_n*\rho_1^{(n)}](\beta)\,,$$

$$2\pi P(y)=M[\phi_l*\rho_1](y)\,,$$

$$2\pi\rho^{(n)}(\theta)=m_nL\cosh\theta+\sum_{m=1}^{l}[\Phi_{nm}*\rho_1^{(m)}](\theta)+[\Phi_n*\rho_1](\theta)\,, \tag{15}$$

where

$$\phi_l(\theta)=1/\cosh(M\theta),\quad \phi_\sigma(\beta)=\frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}\beta}\log\sigma(\beta)\,,$$

$$\Phi_{nm}(\theta)=\frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}\theta}\log S^{(n,m)}(\theta),\quad \Phi_n(\theta)=\frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}\theta}\log S^{(n)}(\theta)\,. \tag{16}$$

The star denotes the rapidity convolution $[\phi*f](\theta)=\int_{-\infty}^{\infty}\phi(\theta-\theta')f(\theta')\,\mathrm{d}\theta'$.

Following the standard procedure of minimizing the free energy $f(R)$ (R=inverse temperature) [11–14], we obtain TBA equations in terms of the following pseudo-energies ε, E, ε_n:

$$\frac{\rho_1}{\rho}=\frac{\mathrm{e}^{-\varepsilon}}{1+\mathrm{e}^{-\varepsilon}},\quad \frac{P_+}{P}=\frac{\mathrm{e}^{-E}}{1+\mathrm{e}^{-E}},\quad \frac{\rho_1^{(n)}}{\rho^{(n)}}=\frac{\mathrm{e}^{-\varepsilon_n}}{1+\mathrm{e}^{-\varepsilon_n}}\,. \tag{17}$$

The TBA equations are

$$Rm\cosh\beta=\theta(\beta)+\frac{1}{2\pi}\left[(\phi_\sigma-[\phi_l*\phi_l])*\ln(1+e^{-\varepsilon})\right](\beta)$$

$$+\frac{M}{2\pi}\left[\phi_l*\ln(1+e^{-E})\right](\beta)+\frac{1}{2\pi}\sum_{n=1}^{l}\left[\Phi_n*\ln(1+e^{-\varepsilon_n})\right](\beta)\,,$$

$$0=E(y)+\frac{M}{2\pi}\left[\phi_l*\ln(1+e^{-\varepsilon})\right](y)\,,$$

$$Rm_n\cosh\theta=\varepsilon_n(\theta)+\frac{1}{2\pi}\left[\Phi_n*\ln(1+e^{-\varepsilon})\right](\theta)+\frac{1}{2\pi}\sum_m\left[\Phi_{nm}*\ln(1+e^{-\varepsilon_m})\right](\theta)\,. \tag{18}$$

In terms of these pseudo-energies, the ground state energy is

$$E(R)=\frac{mR}{\pi}\int_0^\infty d\beta\cosh\beta\ln(1+e^{-\varepsilon(\beta)})+\sum_{n=1}^{l}\frac{m_nR}{\pi}\int_0^\infty d\theta\cosh\theta\ln(1+e^{-\varepsilon_n(\theta)})\,. \tag{19}$$

Performing the standard procedure to evaluate $E(R)$ in terms of the solutions of the TBA equations (18) in the high temperature limit ($R\to 0$) [11–14], we get

$$E(R)\sim-\frac{1}{4\pi R}\left[\int_{\varepsilon(0)}^{\varepsilon(\infty)}d\varepsilon\left(\ln(1+e^{-\varepsilon})+\frac{\varepsilon\,e^{-\varepsilon}}{1+e^{-\varepsilon}}\right)\right.$$

$$\left.+\int_{E(0)}^{E(\infty)}dE\left(\ln(1+e^{-E})+\frac{E\,e^{-E}}{1+e^{-E}}\right)+\sum_{n=1}^{l}\int_{\varepsilon_n(0)}^{\varepsilon_n(\infty)}d\varepsilon_n\left(\ln(1+e^{-\varepsilon_n})+\frac{\varepsilon_n\,e^{-\varepsilon_n}}{1+e^{-\varepsilon_n}}\right)\right]. \tag{20}$$

The pseudo-energies become constant in the region $|\theta|\ll-\ln(mR)$ and can be replaced by the values at $\theta=0$.

Then the TBA equations become pure algebraic equations for the constants $x=e^{-\varepsilon(0)}$, $w=e^{-E(0)}$, and $z_n=e^{-\varepsilon_n(0)}$.

$$x=(1+x)^{\mathcal{N}}(1+w)^{1/2}\prod_{n=1}^{l}(1+z_n)^{\mathcal{N}_n},\quad w=(1+x)^{1/2},\quad z_n=(1+x)^{\mathcal{N}_n}\prod_{m=1}^{l}(1+z_m)^{\mathcal{N}_{nm}}\,. \tag{21}$$

From eq. (18), the first exponent $\mathcal{N}$ is

$$\mathcal{N}=\int_{-\infty}^{\infty}\frac{d\theta}{2\pi}\{\phi_\sigma(\theta)-[\phi_l*\phi_l](\theta)\}\,. \tag{22}$$

Using eq. (6) and the relation [20]

$$\ln\sigma(\theta)=\int_{-\infty}^{\infty}\frac{dz}{\sinh(z-\theta)}\ln[\sigma(z)\sigma(-z)]\,, \tag{23}$$

we find $\mathcal{N}=-\frac{1}{2}l$. The other exponents $\mathcal{N}_n$, $\mathcal{N}_{n,m}$ can be evaluated directly from eq. (5) to be

$$\mathcal{N}_n=-n,\quad \mathcal{N}_{n,m}=[\mathcal{I}(2-\mathcal{I})^{-1}]_{n,m}\,, \tag{24}$$

where $\mathcal{I}$ is the generalized incidence matrix of $A_l^{(2)}$ in the same way as the TBA equations with breathers only [14].

From eq. (18), as the rapidities go to infinite, $\varepsilon(\infty)$ and $\varepsilon_n(\infty)$ become infinite because of the mass terms. For the TBA equations without a driving term (mass term) like the second one in eq. (18), the pseudo-energies like E do not diverge. For the above case, $E(\infty)=0$ ($y=\exp[-E(\infty)]=1$). Using the Rogers dilogarithmic function

$$\mathscr{L}(x)=\tfrac{1}{2}\int_0^x \mathrm{d}t\left(\frac{\ln(1-t)}{t}+\frac{\ln t}{1-t}\right), \tag{25}$$

we can evaluate $E(R)$ with the solutions of the TBA equations. Using eqs. (7), (9) and $E(R)\sim -2\pi/R(\frac{1}{12}c-\Delta_{\min}-\bar{\Delta}_{\min})$ [21], one can derive the following condition to be satisfied if TBA should work for two kinks and an even number (l) of breathers:

$$\frac{1}{12}-\frac{1}{8(4l+1)}=\frac{1}{2\pi^2}\left\{\mathscr{L}\left(\frac{x}{1+x}\right)+\left[\mathscr{L}\left(\frac{w}{1+w}\right)-\mathscr{L}\left(\frac{y}{1+y}\right)\right]+\sum_{n=1}^{l}\mathscr{L}\left(\frac{z_n}{1+z_n}\right)\right\}. \tag{26}$$

We solved eq. (21) using Mathematica™ and checked eq. (26) for a large enough number of l's so that we can conclude that the TBA gives the correct answer; the RSG theory reproduces the correct CFT data of perturbed minimal CFTs $\mathcal{M}_{4/(4l+5)}$. We show numerical results in the first three rows of table 1.

It is possible to generalize the above analysis for any number of kinks. Apart from a difficulty in evaluating

Table 1

N	l	Solutions of eq. (27)	$\frac{1}{12}c-2\Delta_{\min}$
4	2	$x=0.540263$ $w_2=1.24107$ $z_1=0.360892$, $z_2=0.149747$	$\frac{23}{312}$
4	4	$x=0.321259$ $w_2=1.14946$ $z_1=0.343508$, $z_2=0.133784$, $z_3=0.0752709$, $z_4=0.0504372$	$\frac{13}{168}$
4	6	$x=0.228285$ $w_2=1.10828$ $z_1=0.338677$, $z_2=0.12952$, $z_3=0.071017$, $z_4=0.046006$ $z_5=0.0329726$, $z_6=0.0253451$	$\frac{17}{232}$
5	2	$x=0.619914$ $w_2=2.11652$, $w_3=1.76537$ $z_1=0.351153$, $z_2=0.140652$	$\frac{31}{390}$
5	4	$x=0.362677$ $w_2=1.9214$, $w_3=1.70921$ $z_1=0.339933$, $z_2=0.130642$, $z_3=0.0721279$, $z_4=0.0471472$	$\frac{37}{480}$
6	2	$x=0.665401$ $w_2=2.70218$, $w_3=3.3844$, $w_4=2.0939$ $z_1=0.345824$, $z_2=0.135839$	$\frac{3}{38}$
6	4	$x=0.385662$ $w_2=2.42247$, $w_3=3.23507$, $w_4=2.05793$ $z_1=0.337967$, $z_2=0.128937$, $z_3=0.0704515$, $z_4=0.0454328$	$\frac{5}{62}$
7	2	$x=0.693756$ $w_2=3.10099$, $w_3=4.67739$, $w_4=4.33481$, $w_5=2.30972$ $z_1=0.342581$, $z_2=0.132967$	$\frac{37}{462}$
8	2	$x=0.712635$ $w_2=3.381$, $w_3=5.67459$, $w_4=6.35014$, $w_5=5.04146$ $w_6=2.45794$ $z_1=0.340472$, $z_2=0.131121$	$\frac{97}{1200}$

the eigenvalues of the RSOS transfer matrix, the derivation of the TBA equations is almost the same as before. If the number of kinks increases, the RSOS sector should contain more "pseudo-particles" which act as massless particles because contraint equations like (14) have more solutions which have different imaginary components for the same real part [19]. This leads to the conjecture given in ref. [16] that the exponents in kink–kink parts of the TBA equations are given by an A_{N-2} incidence matrix. This conjecture must be equally valid for our case because the quantum group structures of the two cases are exactly the same provided the number of species of breathers is even. The only difference is the exponent $\mathcal{N}$ which is 0 for the case of no breather [16]. For the more general case, $\mathcal{N}=-\frac{1}{2}l$ as we showed in eq. (22) ($\phi_l(\theta)=1/\cosh(4\theta/P)$).

Including the pseudo-energies E_a of the "pseudo-particles" ($w_a=\mathrm{e}^{-E_a(0)}$), the most general equations for the pseudo-energies at zero rapidity are given by

$$x=(1+x)^{-l/2}\prod_{a=2}^{N-2}(1+w_a)^{L_{1a}/2}\prod_{n=1}^{l}(1+z_n)^{\mathcal{N}_n}, \qquad w_a=(1+x)^{L_{1a}/2}\prod_{b=2}^{N-2}(1+w_b)^{L_{ab}/2}, \quad a=2,3,\ldots,$$

$$z_n=(1+x)^{\mathcal{N}_n}\prod_{m=1}^{l}(1+z_m)^{\mathcal{N}_{nm}}, \tag{27}$$

where L_{ab} are the elements of the A_{N-2} incidence matrix, and at infinite rapidity $y_a=\mathrm{e}^{-E_a(\infty)}$ should satisfy

$$y_a=\prod_{b=2}^{N-3}(1+y_b)^{\bar{L}_{ab}/2}, \quad a=2,3,\ldots,N-3, \tag{28}$$

in terms of the reduced incidence matrix of A_{N-3}, $\bar{L}_{ab}$. Eq. (26) is changed to

$$\frac{1}{12}-\frac{1}{N(Nl+N+1)}=\frac{1}{2\pi^2}\left[\mathcal{L}\left(\frac{x}{1+x}\right)+\sum_{a=2}^{N-2}\mathcal{L}\left(\frac{w_a}{1+w_a}\right)-\sum_{a=2}^{N-3}\mathcal{L}\left(\frac{y_a}{1+y_a}\right)+\sum_{n=1}^{l}\mathcal{L}\left(\frac{z_n}{1+z_n}\right)\right]. \tag{29}$$

We have checked eq. (29) using Mathematica™ for several cases (see table 1 for numerical results). While the solutions of eq. (27) are not simple numbers, it is remarkable that the sums of the Rogers dilogarithmic functions reproduce the rational values of the left-hand side of eq. (29) correctly. This may mean that there exists a new kind of sum rules. This concludes that minimal CFT $\mathcal{M}_{p/q}$ ($p=N$, $q=Nl+N+1$, l even) perturbed by $\Phi_{1,3}$ is the RSG$[p/q]$ theory that has $N-2$ kinks and l breathers in the spectrum.

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