Nonrelativistic factorizable scattering theory and the Calogero-Sutherland model

Changrim Ahn and Kong-Ju-Bock Lee
Department of Physics, Ewha Womans University, Seoul 120-750, Korea

Soonkeon Nam
Department of Physics, Kyung Hee University, Seoul 130-701, Korea

(Received 4 March 1996)

We solve the SU(N)-invariant Yang-Baxter equations imposing only the unitarity condition. The usual S matrices should satisfy the crossing symmetry which originates from the CPT invariance of relativistic quantum-field theory. In this paper, we consider nonrelativistic SU(N)-invariant factorizable S matrices by relaxing the crossing symmetry and making the amplitudes for creating and annihilating new particles vanish and find that these S matrices are exactly the same as those of the multicomponent Calogero-Sutherland model, the quantum-mechanical model with the hyperbolic potential between particles and antiparticles. This particular solution is of interest since it cannot be obtained as a nonrelativistic limit of any known relativistic solutions of the SU(N)-invariant Yang-Baxter equations. [S1050-2947(96)06011-8]

PACS number(s): 03.65.Nk, 11.55.Ds

Integrability plays an important role in understanding nonperturbative aspects of various models ranging from the field theoretical and statistical models to nonrelativistic quantum-mechanical many-body systems. If the number of conserved charges are equal to the degrees of freedom, the number of particles and their momenta are preserved and multiparticle interaction can be factorized into products of two-body interactions. A consistency condition for this factorization is the Yang-Baxter equation (YBE).

The YBE can determine the S matrices of the integrable quantum-field theories and the Boltzmann weights of equilibrium statistical mechanics models in two dimensions. To fix the S matrices completely, one should impose the unitarity and the crossing symmetry along with the particle spectrum. The unitarity is the conservation law of the probability while the crossing symmetry arises from the CPT symmetry of the relativistic quantum-field theories.

In this paper, we find scattering amplitudes of a quantum-field theory and a nonrelativistic quantum mechanics and relate these solutions. The S matrices of the quantum-field theory are derived by solving the SU(N)-invariant YBE and imposing only the unitarity condition while reserving the crossing symmetry. Then we analyze these solutions in the following manner. To get the nonrelativistic quantum-mechanical limit, we let the rapidities be small and the amplitudes for the creation and the annihilation processes of the particles and antiparticles vanish. In this way, we can find a complete class of solutions which include the relativistic S matrices as a subset along with genuinely nonrelativistic solutions.

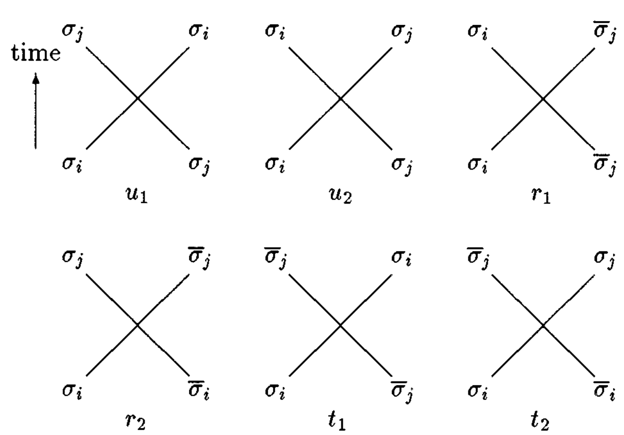

FIG. 1. Scattering amplitudes of particles and antiparticles with colors.

Our main result is that one of the genuinely nonrelativistic solutions of the SU(N)-invariant YBE is identical to scattering amplitudes of the multicomponent Calogero-Sutherland model (CSM) [1,2]. It is remarkable that this correspondence cannot be obtained by taking a merely nonrelativistic limit of the relativistic (CPT-invariant) solutions.

Recently, there has been a great interest in the CSM, a nonrelativistic quantum-mechanical n-body system with long-ranged two-body potentials. The CSM is closely related to the integrable spin chains with long-ranged interactions [3,4], random matrix theory [5], and fractional statistics [6]. Furthermore, this model is also connected with a field theory. The scattering amplitudes of the CSM can be derived by simply taking a nonrelativistic limit of the sine-Gordon soliton S matrix [7,8].[1] In this paper we generalize this correspondence to the multicomponent CSM, the n-body quantum-mechanical system of colored particles and antiparticles interacting via integrable long-ranged potential of hyperbolic type and nonrelativistic factorizable S-matrix theory with SU(N) invariance. However, our result is different from that of the sine Gordon in the sense that the S matrix of the CSM is not derived from the nonrelativistic limit of the relativistic scattering theory solved completely in [10].

We start with the SU(N)-invariant YBE and the unitarity condition. This can be represented pictorially as in Fig. 1. u's denote the scattering amplitudes between particles (or

[1]This connection was also observed for the boundary sine-Gordon equation in its relation to BC_n type CSM [9].

TABLE I. Complete solutions of nonrelativistic SU(N) invariant Yang-Baxter equations.

Class	$u_1(\theta)$	$u_2(\theta)$	$r_1(\theta)$	$r_2(\theta)$	$t_1(\theta)$	$t_2(\theta)$
I	$\frac{\theta}{\gamma\pm\theta}U(\theta)$	$\frac{\gamma}{\theta}u_1(\theta)$	0	0	$T(\theta)$	0
II	$\frac{\theta}{\gamma\pm\theta}U(\theta)$	$\frac{\gamma}{\theta}u_1(\theta)$	0	0	$T(\theta)$	$-\frac{\gamma}{\frac{\gamma N}{2}+\theta}t_1(\theta)$
III	$t_1(\theta)$	$r_1(\theta)$	$\frac{\gamma}{\theta}t_1(\theta)$	$\frac{\gamma}{\gamma(1-N)-\theta}t_1(\theta)$	$\frac{\theta}{\gamma+\theta}U(\theta)$	$r_2(\theta)$
IV	$-t_1(\theta)$	$r_1(\theta)$	$-\frac{\gamma}{\theta}t_1(\theta)$	$\frac{\gamma}{-\gamma(1+N)-\theta}t_1(\theta)$	$\frac{\theta}{\gamma-\theta}U(\theta)$	$r_2(\theta)$
V	0	$r_1(\theta)$	$R(\theta)$	$r_1(\theta)$	0	$r_1(\theta)$
VI	0	$e^{\gamma\theta}r_1(\theta)$	$R(\theta)$	$-\frac{N(e^{2\gamma\theta}-1)}{N^2e^{2\gamma\theta}-1}r_1(\theta)$	0	$N^{-1}e^{-\gamma\theta}r_2(\theta)$
VII	$t_1(\theta)$	$r_1(\theta)$	$-\frac{\gamma}{\theta}t_1(\theta)$	0	$\frac{-\theta}{\gamma+\theta}U(\theta)$	0
VIII	$\frac{\theta}{\gamma\pm\theta}U(\theta)$	$\frac{\gamma}{\theta}u_1(\theta)$	$R(\theta)$	0	0	0
IX	0	$U(\theta)$	$R(\theta)$	0	0	$\gamma\left[U(\theta)-U(-\theta)\frac{R(\theta)}{R(-\theta)}\right]$
X	0	$U(\theta)$	$r_1(\theta)$	0	$\gamma\left[U(\theta)-U(-\theta)\frac{r_1(\theta)}{r_1(-\theta)}\right]$	0 or $-\frac{N}{2}t_1(\theta)$

antiparticles), r's the reflected one, and t's the transmitted one between the particle and the antiparticle.

The YBE and the unitarity condition consist of 15 equations for these amplitudes, and seven of them include neither r_2 nor t_2. Note that r_2 and t_2 represent the amplitudes of pair-creation and pair-annihilation processes. The seven equations are the following:

$$u_1(\theta)u_1(-\theta)+u_2(\theta)u_2(-\theta)=1, \tag{1}$$

$$u_1(\theta)u_2(-\theta)+u_2(\theta)u_1(-\theta)=0,$$

$$t_1(\theta)t_1(-\theta)+r_1(\theta)r_1(-\theta)=1,$$

$$t_1(\theta)r_1(-\theta)+r_1(\theta)t_1(-\theta)=0,$$

$$u_1(\theta)t_1(\theta+\theta')r_1(\theta')=t_1(\theta)u_1(\theta+\theta')r_1(\theta'),$$

$$u_2(\theta)t_1(\theta+\theta')r_1(\theta')=r_1(\theta)r_1(\theta+\theta')t_1(\theta')+t_1(\theta)u_2(\theta+\theta')r_1(\theta'),$$

$$u_2(\theta)u_1(\theta+\theta')u_2(\theta')=u_1(\theta)u_2(\theta+\theta')u_2(\theta')+u_2(\theta)u_2(\theta+\theta')u_1(\theta'),$$

where θ is the spectral parameter. The ''minimal'' solution of these equations is

$$u_1(\theta)=t_1(\theta)=\frac{-\theta}{\gamma+\theta}\frac{\Gamma(\theta)\Gamma(\gamma-\theta)}{\Gamma(-\theta)\Gamma(\gamma+\theta)},$$

$$u_2(\theta)=r_1(\theta)=\frac{\gamma}{\gamma+\theta}\frac{\Gamma(\theta)\Gamma(\gamma-\theta)}{\Gamma(-\theta)\Gamma(\gamma+\theta)}, \tag{2}$$

where γ is the arbitrary parameter. Plugging this minimal solution into the rest of the equations, we obtain all possible solutions of r_2 and t_2. Among these, those with $r_2=t_2=0$ are particularly interesting since they can be related to the nonrelativistic quantum-mechanical systems. Indeed, replacing θ with ik and interpreting the parameter γ as the coefficient of the hyperbolic interaction λ in CSM, we will show that these scattering amplitudes are identical to those of the multicomponent CSM.

We list the complete solutions of SU(N)-invariant YBE without the crossing symmetry in Table I. $U(\theta)$ satisfy $U(\theta)U(-\theta)=1$ due to the unitarity and similarly for $R(\theta)$, and $T(\theta)$. It is easy to show that the classes I–VI are exactly the nonrelativistic limits of the relativistic solutions by Berg *et al.* [10], i.e., imposing the crossing symmetry of $u_1(\theta)=t_1(i\pi-\theta)$, $u_2(\theta)=t_2(i\pi-\theta)$, and $r_1(\theta)=r_2(i\pi-\theta)$, which fixes the parameter γ to a certain value for each classes. Class I describes a system without reflection, pair creantion, and pair annihilation. Thus it is a good candidate for a nonrelativistic quantum-mechanical system. γ is fixed to zero with the crossing symmety. Class II does not have reflecting amplitudes and $\gamma=-2\pi i/N$ with the crossing relation. It is related to the CP^{N-1} model SU(N) chiral Gross-Neveu model. Class III includes the $O(2N)$ ($N>1$) symmetric scattering matrices obtained by Zamolodchikov and Zamolodchikov [8]. The crossing symmetry fixes γ to $i\pi/(1-N)$. Class IV has a similar structure to class III with $\gamma=-i\pi/(1+N)$ in the relativistic limit. Classes V and VI are the other nontrivial solutions with a crossing relation. Classes IX and X do not allow any nontrivial solutions with a crossing relation. No models are associated with these classes yet.

Classes VII and VIII are the most interesting since they have vanishing r_2 and t_2. Since these two classes are identical under the interchange of t_1 with r_1, we will concentrate on Class VII only and will discuss how it can be related to the multicomponent CSM.

We begin with the multicomponent CSM where the Hamiltonian is given by

$$H = -\sum_{i}^{n} \frac{\partial^2}{\partial x_i^2} + \sum_{i<j} \frac{\lambda(\lambda + P_{ij})}{\sinh^2(x_i - x_j)}, \tag{3}$$

where P_{ij} denotes the exchange operator of the colors of (anti-)particles at x_i and x_j. This model has been shown to be integrable by several authors [11,12]. Since $P_{ij}^2 = I$, we can define the eigenstates of P_{ij} as $|\pm\rangle = (1/\sqrt{2})(|\sigma_i\sigma_j\rangle \pm |\sigma_j\sigma_i\rangle)$ such that $P_{ij}|\pm\rangle = \pm|\pm\rangle$.

To obtain the two-body S matrix, we consider the scattering eigenstates of the Schrödinger equation $[-(d^2/dx^2) + \{\lambda(\lambda+1)/\sinh^2 x\}]\psi_k(x) = k^2\psi_k(x)$. Due to the underlying SU(1,1) structure of the scattering problem [13], $\psi_k(x)$ is proportional to $(\sinh x)^{\lambda+1}{}_2F_1[(\lambda+1+ik)/2, (\lambda+1-ik)/2, \lambda+3/2; -\sinh^2 x]$, where ${}_2F_1$ is the hypergeometric function. The asymptotic states for $x \to \infty$ are

$$\psi_k(x) \to C\left(e^{ikx} \frac{\Gamma(ik)\Gamma(2\lambda+2)}{\Gamma(\lambda+1+ik)\Gamma(\lambda+1)} + e^{-ikx} \frac{\Gamma(-ik)\Gamma(2\lambda+2)}{\Gamma(\lambda+1-ik)\Gamma(\lambda+1)} \right) \tag{4}$$

and the two-body scattering matrices are

$$S^+(k) = \frac{\Gamma(ik)\Gamma(1+\lambda-ik)}{\Gamma(-ik)\Gamma(1+\lambda+ik)},$$

$$S^-(k) = \frac{\Gamma(ik)\Gamma(\lambda-ik)}{\Gamma(-ik)\Gamma(\lambda+ik)}, \tag{5}$$

for $P_{ij} = \pm 1$, respectively [14,15].

Now returning to the $|\sigma_i\sigma_j\rangle$ basis, one can obtain particle-particle scattering amplitudes straightforwardly as follows:

$$S_{\sigma_i\sigma_j}^{\sigma_j\sigma_i} \equiv u_1 = \tfrac{1}{2}(S^+ + S^-), \quad S_{\sigma_i\sigma_j}^{\sigma_i\sigma_j} \equiv u_2 = \tfrac{1}{2}(S^+ - S^-), \tag{6}$$

for $\sigma_i \neq \sigma_j$. Note that $S_{\sigma_i\sigma_i}^{\sigma_i\sigma_i} = u_1 + u_2$ if $\sigma_i = \sigma_j$. Similarly, the scattering amplitudes of particles and antiparticles where $\sigma_i \neq \overline{\sigma_j}$ become

$$S_{\sigma_i\overline{\sigma_j}}^{\sigma_i\overline{\sigma_j}} \equiv r_1 = \tfrac{1}{2}(S^+ + S^-), \quad S_{\sigma_i\overline{\sigma_j}}^{\overline{\sigma_j}\sigma_i} \equiv t_1 = \tfrac{1}{2}(S^+ - S^-), \tag{7}$$

where $\overline{\sigma_j}$ stands for the color of an antiparticle. (See Fig. 1 for schematic definitions of the amplitudes.)

The scattering amplitudes of the same color have to be dealt with some care. The most general eigenstates of P_{ij} are now of the form $|\pm\rangle = (1/\sqrt{2})(|A\rangle \pm |B\rangle)$, where $|A\rangle = \mathcal{N}\Sigma_{i=1}^N a_i|\sigma_i\overline{\sigma_i}\rangle$ and $|B\rangle = \mathcal{N}\Sigma_{i=1}^N a_i|\overline{\sigma_i}\sigma_i\rangle$ for some coefficients a_i's and $\mathcal{N} = (\Sigma_{i=1}^N a_i^2)^{-1/2}$. The scattering amplitudes $S^\pm$ in the basis of $|\pm\rangle$ can be related to those in the basis of $|\sigma_i\overline{\sigma_i}\rangle$ as follows:

$$\tfrac{1}{2}(S^+ + S^-) = r_1 + Mr_2, \quad \tfrac{1}{2}(S^+ - S^-) = t_1 + Mt_2, \tag{8}$$

where $M = \mathcal{N}^2\Sigma_{i,j}^N a_i a_j$. Comparing Eq. (5) with (6), we find $r_2 = t_2 = 0$. With $S^\pm$ given in Eq. (3), the scattering amplitudes of the SU(N)-invariant CSM yield

$$u_1 = t_1 = \frac{-ik}{\lambda+ik} \frac{\Gamma(ik)\Gamma(\lambda-ik)}{\Gamma(-ik)\Gamma(\lambda+ik)},$$

$$u_2 = r_1 = \frac{\lambda}{\lambda+ik} \frac{\Gamma(ik)\Gamma(\lambda-ik)}{\Gamma(-ik)\Gamma(\lambda+ik)},$$

$$r_2 = t_2 = 0. \tag{9}$$

Indeed these amplitudes belong to class VII of the YBE.

In this paper, we used the YBE with the unitarity as a unified tool to derive the S matrices of nonrelativistic many-particle systems as well as those of relativistic field theories. Additional constraints select out the corresponding solutions. For the relativistic cases, the crossing symmetry is required due to the CPT invariance. For the nonrelativistic cases, on the other hand, a natural assumption of the vanishing amplitudes of creation and annihilation is necessary. An interesting result is that we derived the S matrices of the multicomponent CSM where the potentials between two particles are all $1/\sinh^2 x$ types irrespective of the species from the SU(N)-invariant YBE. We want to point out that there exists a more general exactly solvable potential (Pöschl-Teller) [13] which contains both $1/\sinh^2 x$ and $1/\cosh^2 x$ [16]. It has been also known that the $O(2)$-invariant scattering theory (the sine-Gordon model) is related to the nonrelativistic Hamiltonian system with the $1/\sinh^2 x$ potential for (anti-)particle-(anti-)particle scattering [8] and the $1/\cosh^2 x$ for particle-antiparticle [7]. It would be interesting to consider the case where particle-antiparticle scattering potential is different from particle-particle potential with particles carrying colors. We would like to emphasize the approach to the multicomponent CSM and the generalized Haldane-Shastry model [3,4,17] based on the factorizable S-matrix theory can be promising. We hope our approach can be generalized to other integrable Hamiltonian systems.

We would like to thank T. Deguchi, P. Fendley, N. Kawakami, C. Lee, V. Pasquier, C. Rim, and R. Sasaki for helpful discussion. This work is supported in part by the Korea Science and Engineering Foundation under Contracts No. 95-0701-04-01-3 (C.A. and K.L.) and 961-0201-006-2 (C.A.), and the Ministry of Education under Contracts No. BSRI-96-2428 (C.A.), BSRI-96-2428 (K.L.), and BSRI-96-2442 (S.N.). S.N. also acknowledges the Korea-Japan Exchange program of KOSEF-JSPS. We also acknowledge the partial support of the Center for Theoretical Physics at Seoul National University.

[1] P. Calogero, J. Math. Phys. **10**, 2191 (1969).
[2] B. Sutherland, Phys. Rev. A **5**, 1372 (1972).
[3] F.D.M. Haldane, Phys. Rev. Lett. **60**, 635 (1988).
[4] B.S. Shastry, Phys. Rev. Lett. **60**, 639 (1988).
[5] B.D. Simons, P.A. Lee, and B.L. Altshuler, Phys. Rev. Lett. **72**, 64 (1994).
[6] F.D.M. Haldane, Phys. Rev. Lett. **67**, 937 (1991).
[7] V. Korepin, JETP Lett. **23**, 201 (1976).
[8] A.B. Zamolodchikov and Al.B. Zamolodchikov, Ann. Phys. (N.Y.) **120**, 253 (1980).
[9] A. Kapustin and S. Skorik, Phys. Lett. A **196**, 47 (1994).
[10] B. Berg, M. Karowski, P. Weisz, and V. Kurak, Nucl. Phys. **B134**, 125 (1978).
[11] A.P. Polychronakos, Phys. Rev. Lett. **69**, 703 (1992).
[12] Z.N.C. Ha and F.D.M. Haldane, Phys. Rev. B **46**, 9359 (1992).
[13] Y. Alhassid, F. Gürsey, and F. Iachello, Ann. Phys. (N.Y.) **148**, 346 (1983).
[14] M.A. Olshanetsky and A.M. Perelomov, Phys. Rep. **94**, 313 (1983).
[15] B. Sutherland and B.S. Shastry, Phys. Rev. Lett. **71**, 5 (1993).
[16] B. Sutherland and R.A. Römer, Phys. Rev. Lett. **71**, 2789 (1993).
[17] N. Kawakami, Phys. Rev. B **46**, 1005 (1992).