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Applications of Reflection Amplitudes in Toda-Type Theories[1]

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Reflection amplitudes are defined as two-point functions of certain class of conformal field theories where primary fields are given by vertex operators with real couplings. Among these, we consider (Super-) Liouville theory and simply and non-simply laced Toda theories. In this paper we show how to compute the scaling functions of effective central charge for the models perturbed by some primary fields which maintains integrability. This new derivation of the scaling functions are compared with the results from conventional TBA approach and confirms our approach along with other non-perturbative results such as exact expressions of the on-shell masses in terms of the parameters in the action, exact free energies. Another important application of the reflection amplitudes is a computation of one-point functions for the integrable models. Introducing functional relations between the one-point functions in terms of the reflection amplitudes, we obtain explicit expressions for simply-laced and non-simply-laced affine Toda theories. These nonperturbative results are confirmed numerically by comparing the free energies from the scaling functions with exact expressions we obtain from the one-point functions.

KEY WORDS: Reflection amplitude; affine Toda field theory; conformal field theory; super-Liouville theory; thermodynamic Bethe ansatz; one-point function.

1. INTRODUCTION

There is a large class of 2D quantum field theories (QFTs) which can be considered as perturbed conformal field theories (CFTs).[(1)] These theories

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are completely defined if one specifies its CFT data and the relevant operator which plays the role of perturbation. The CFT data contain explicit information about ultraviolet (UV) asymptotics of the field theory while its long distance property is the subject of analysis. If a perturbed CFT contains only massive particles, it is equivalent to the relativistic scattering theory and is completely defined by specifying the S-matrix. Contrary to CFT data the S-matrix data exibit some information about long distance properties of the theory in an explicit way, while the UV asymptotics have to be derived.

A link between these two kinds of data would provide a good view point for understanding the general structure of 2D QFTs. In general this problem does not look tractable. Whereas the CFT data can be specified in a relatively simple way, the general S-matrix is very complicated object even in 2D. However, there exists an important class of 2D QFTs (integrable theories) where scattering theory is factorized and S-matrix can be described in great details.

In this case one can apply the nonperturbative methods based on the S-matrix data. One of these methods is thermodynamic Bethe ansatz (TBA).[2, 3] It gives the possibility to calculate the ground state energy $E(R)$ (or effective central charge $c_{\text{eff}}(R)$) for the system on the circle of size R. At small R the UV asymptotics of $c_{\text{eff}}(R)$ can be compared with that following from the CFT data.

Usually the UV asymptotics for the effective central charge can be derived from the conformal perturbation theory. In this case the corrections to $c_{\text{eff}}(0)=c_{\text{CFT}}$ have a form of series in R^{γ} where γ is defined by the dimension of perturbing operator. However, there is an important class of QFTs where the UV asymptotics of $c_{\text{eff}}(R)$ is mainly determined by the zero-mode dynamics (see for example, refs. 4–7). In this case the UV corrections to c_{CFT} have the form of series in inverse powers of $\log(1/R)$. This UV expansion is also encoded in CFT data.[6]

The simplest integrable QFT with the logarithmic expansion for the effective central charge is the sinh-Gordon (ShG) model, which is an integrable deformation of Liouville conformal field theory (LFT). It was shown in paper ref. 6 that the crucial role in the description of the zero-mode dynamics in the ShG model is played by the "reflection amplitude" of the LFT, which determines the asymptotics of the ground state wave function in this theory. (The reflection amplitudes in CFT define the linear transformations between different exponential fields, corresponding to the same primary field of chiral algebra.)

In this paper, we want to show that these zero-modes dynamics are quite general features of 2D QFTs with exponential interactions. Imposing the integrability, we will consider the Bullough–Dodd model which is another integrable perturbation of the LFT, supersymmetric ShG models,[7] and affine Toda field theories (ATFTs) associated with both simply laced and non-simply laced Lie algebras.[8, 9] Each model shows quite unique properties in the S-matrix data; some has non-diagonal S-matrix, some has many particles with different masses, and so on. In this review of our recent works,[7–10] we want to describe UV asymptotic behaviours of these "Toda-type" models, i.e., a class of the integrable QFTs with exponential interactions, in unifying way in terms of the zero-mode dynamics. The perturbing term in the model restricts the zero-mode dynamics to a box of size $l \sim \log(1/R)$ in the auxiliary space with dimension equal to the number of independent zero-modes. This leads to the quantization condition for the momentum P conjugated to the zero-modes and the solution $P(R)$ determines all logarithmic terms in the UV asymptotics of the effective central charge $c_{\rm eff}(R)$. Although it may seem quite model-dependent, it turns out that the basic dynamics in common is the reflection of a zero-mode off the interacting potential of the LFT. In all cases the results agree perfectly with TBA results based on the S-matrix data. The remarkable feature is that effective central charge calculated from the CFT data with subtracted bulk free energy term (like in TBA approach) gives a good agreement with the TBA results even outside the UV region (at $R \sim \mathcal{O}(1)$). This "empirical" fact still needs the explanation.

Finally the reflection amplitudes can be used[11–13] to find the exact one-point functions of this class of integrable models. One needs only symmetric properties of the given Lagrangian and analytic properties of the one-point function. Explicit calculation is given for various models.[6, 11–15, 8, 9]

In the Section 2, we introduce the reflection amplitudes for the CFTs we are interested in. These amplitudes are interpreted as quantum mechanical reflections off the potential wall of the zero-modes dynamics in Section 3. In Section 4 we analyze the off-critical integrable models. Due to the integrable perturbations, the zero-modes are confined in the potential well and the conjugate momenta are quantized. Using this, we calculate the UV asymptotics for the effective central charges. In Section 5 we compare this asymptotics with numerical solutions of TBA equations. We derive the exact one-point functions and free energies in Section 6.

2. NORMALIZATION FACTORS AND REFLECTION AMPLITUDES

In this section, we introduce the reflection amplitudes for the Toda-type CFTs with background charges.

2.1. Toda-Type CFTs

We start with non-affine Toda theories (NATTs) whose actions are given by

$$\mathscr{A}=\int d^2x\left[\frac{1}{8\pi}(\partial_\mu\boldsymbol{\varphi})^2+\sum_{i=1}^{r}\mu_i e^{b\mathbf{e}_i\cdot\boldsymbol{\varphi}}\right] \tag{1}$$

where $\mathbf{e}_i$, $i=1,\dots,r$ are the simple roots of the Lie algebra G of rank r. For simply laced algebras, the μ_i's are all the same as μ. Non-simply laced ATFTs have standard simple roots with $\mathbf{e}_i^2=2$ and nonstandard simple roots with $\mathbf{e}_i^2\equiv\xi^2(\neq 2)$. We choose the corresponding parameters μ_i as μ (for standard roots) and μ' (for nonstandard ones), respectively.[6]

The simplest case is the A_1 NATT, or the LFT with an action

$$\mathscr{A}=\int d^2x\left[\frac{1}{8\pi}(\partial_\mu\varphi)^2+\mu e^{\sqrt{2}\,b\varphi}\right] \tag{2}$$

With appropriate background charges, these Toda-type QFTs are the CFTs. To describe the generator of conformal symmetry we introduce the complex coordinates $z=x_1+ix_2$ and $\bar{z}=x_1-ix_2$ and vector:

$$\mathbf{Q}=b\boldsymbol{\rho}+\frac{1}{b}\boldsymbol{\rho}^\vee,\qquad \boldsymbol{\rho}=\frac{1}{2}\sum_{\boldsymbol{\alpha}>0}\boldsymbol{\alpha},\qquad \boldsymbol{\rho}^\vee=\frac{1}{2}\sum_{\boldsymbol{\alpha}>0}\boldsymbol{\alpha}^\vee \tag{3}$$

where the sum in definition of Weyl vector $\boldsymbol{\rho}$ ($\boldsymbol{\rho}^\vee$) runs over all positive roots $\boldsymbol{\alpha}$ (co-roots $\boldsymbol{\alpha}^\vee$) of G.

The holomorphic stress-energy tensor

$$T(z)=-\tfrac{1}{2}(\partial_z\boldsymbol{\varphi})^2+\mathbf{Q}\cdot\partial_z^2\boldsymbol{\varphi} \tag{4}$$

ensures the local conformal invariance of the NATT with the central charge $c=r+12\mathbf{Q}^2$. For the simply laced algebras, these expressions are simplified due to $\mathbf{Q}=Q\boldsymbol{\rho}$ with $Q=b+1/b$ that the central charge is given by $c=r(1+h(h+1)\,Q^2)$ (h is Coxeter number).

Another model we are interested in is the $N=1$ supersymmetric LFT with an action

$$\mathscr{A}=\int d^2x\,\frac{1}{8\pi}(\partial_a\phi)^2-\frac{1}{2\pi}(\bar{\psi}\,\partial\bar{\psi}+\psi\,\bar{\partial}\psi)+i\mu b^2\psi\bar{\psi}e^{b\phi}+\frac{\pi\mu^2b^2}{2}\,e^{2b\phi} \tag{5}$$

[6] We choose the convention that the length squared of the long roots are four for $C_r^{(1)}$ and two for the other untwisted algebras.

The super-LFT with the background charge $Q=b+1/b$ has the central charge $c_{SL}=3/2(1+2Q^2)$.

2.2. Reflection Amplitudes

2.2.1. NATTs

Besides the conformal invariance the NATT possesses extended symmetry generated by $W(G)$-algebra. The full chiral $W(G)$-algebra contains r holomorphic fields $W_j(z)$ ($W_2(z)=T(z)$) with spins j which follows the exponents of Lie algebra G. The primary fields Φ_w of $W(G)$ algebra are classified by r eigenvalues w_j, $j=1,..., r$ of the operator $W_{j,0}$ (the zeroth Fourier component of the current $W_j(z)$):

$$W_{j,0}\Phi_w=w_j\Phi_w, \qquad W_{j,n}\Phi_w=0, \qquad n>0 \tag{6}$$

The exponential fields

$$V_{\boldsymbol{a}}(x)=e^{(\mathbf{Q}+\boldsymbol{a})\cdot\boldsymbol{\varphi}(x)} \tag{7}$$

are spinless conformal primary fields with dimensions $\Delta(\boldsymbol{a})=w_2(\boldsymbol{a})=(\mathbf{Q}^2-\boldsymbol{a}^2)/2$. The fields $V_{\boldsymbol{a}}$ are also primary with respect to all chiral algebra $W(G)$ with the eigenvalues w_j depending on $\boldsymbol{a}$. The functions $w_j(\boldsymbol{a})$, which define the representation of $W(G)$-algebra possess the symmetry with respect to the Weyl group $\mathscr{W}$ of Lie algebra G,[16, 17] i.e., $w_j(\hat{s}\boldsymbol{a})=w_j(\boldsymbol{a})$; for any $\hat{s}\in\mathscr{W}$. It means that the fields $V_{\hat{s}\boldsymbol{a}}$ for different $\hat{s}\in\mathscr{W}$ are reflection images of each other and are related by the linear transformation:

$$V_{\boldsymbol{a}}(x)=R_{\hat{s}}(\boldsymbol{a})\ V_{\hat{s}\boldsymbol{a}}(x) \tag{8}$$

where $R_{\hat{s}}(\boldsymbol{a})$ is the "reflection amplitude." This function is an important object in CFT and plays a crucial role in the calculation of the one-point functions in perturbed CFT.[19]

To calculate the function $R_{\hat{s}}(\boldsymbol{a})$ for simply laced NATTs, we introduce the fields Φ_w:

$$\Phi_w(x)=N^{-1}(\boldsymbol{a})\ V_{\boldsymbol{a}}(x) \tag{9}$$

where normalization factor $N(\boldsymbol{a})$ is chosen in the way that field Φ_w satisfies the conformal normalization condition

$$\langle\Phi_w(x)\,\Phi_w(y)\rangle=\frac{1}{|x-y|^{4\Delta}} \tag{10}$$

The normalized fields Φ_w are invariant under reflection transformations and hence;

$$R_{\hat{s}}(\boldsymbol{a}) = \frac{N(\boldsymbol{a})}{N(\hat{s}\boldsymbol{a})} \tag{11}$$

For the calculation of the normalization factor $N(\boldsymbol{a})$, we can use the integral representation for the correlation functions of the $W(G)$-invariant CFT. (See ref. 16 for details.) We note that operators $\hat{Q}_i$ defined as

$$\hat{Q}_i = \mu \int d^2x \, e^{b\mathbf{e}_i \cdot \boldsymbol{\varphi}(x)} \tag{12}$$

commute with all of the elements of $W(G)$-algebra and can be used as screening operators for the calculation of the correlation functions in the NATT. If parameters $\boldsymbol{a}$ satisfy the condition

$$2\mathbf{Q} + 2\boldsymbol{a} + \sum_{i=1}^{r} k_i \mathbf{e}_i = 0 \tag{13}$$

with non-negative integer k_i, we obtain from Eqs. (9) and (10) the following expression for the function $N(\boldsymbol{a})$ in terms of Coulomb integrals:[16]

$$N^2(\boldsymbol{a}) = |x|^{4\Delta} \left\langle V_{\boldsymbol{a}}(x)\, V_{\boldsymbol{a}}(0) \prod_{i=1}^{r} \frac{\hat{Q}_i^{k_i}}{k_i!} \right\rangle \tag{14}$$

where the expectation value in Eq. (14) is taken over the Fock vacuum of massless fields $\boldsymbol{\varphi}$ with the correlation functions

$$\langle \varphi_a(x)\, \varphi_b(y) \rangle = -\delta_{ab} \log |x-y|^2$$

The normalization integral can be calculated and the result has the form:

$$N^2(\boldsymbol{a}) = (\pi\mu\gamma(b^2))^{-2\boldsymbol{\rho}\cdot(\mathbf{Q}+\boldsymbol{a})/b} \times \prod_{\boldsymbol{\alpha}>0} \frac{\Gamma(1+Q_{\boldsymbol{\alpha}}/b)\,\Gamma(1+Q_{\boldsymbol{\alpha}}b)\,\Gamma(1+a_{\boldsymbol{\alpha}}/b)\,\Gamma(1+a_{\boldsymbol{\alpha}}b)}{\Gamma(1-Q_{\boldsymbol{\alpha}}/b)\,\Gamma(1-Q_{\boldsymbol{\alpha}}b)\,\Gamma(1-a_{\boldsymbol{\alpha}}/b)\,\Gamma(1-a_{\boldsymbol{\alpha}}b)} \tag{15}$$

in terms of the scalar products

$$Q_{\boldsymbol{\alpha}} = \mathbf{Q}\cdot\boldsymbol{\alpha}, \qquad a_{\boldsymbol{\alpha}} = \boldsymbol{a}\cdot\boldsymbol{\alpha} \tag{16}$$

where the product runs over all positive roots of Lie algebra G.

We accept Eq. (15) as the proper analytical continuation of the function $N^2(\boldsymbol{a})$ for all $\boldsymbol{a}$. It gives us the following expression for the reflection amplitude $R_{\hat{s}}(\boldsymbol{a})$:

$$R_{\hat{s}}(\boldsymbol{a}) = \frac{N(\boldsymbol{a})}{N(\hat{s}\boldsymbol{a})} = \frac{A_{\hat{s}\boldsymbol{a}}}{A_{\boldsymbol{a}}} \tag{17}$$

where

$$A_{\boldsymbol{a}} = (\pi\mu\gamma(b^2))^{\boldsymbol{\rho}\cdot\boldsymbol{a}/b} \prod_{\boldsymbol{\alpha}>0} \Gamma(1-a_{\boldsymbol{\alpha}}/b)\,\Gamma(1-a_{\boldsymbol{\alpha}}b) \tag{18}$$

For non-simply laced NATTs, the expression is generalized to

$$A_{\boldsymbol{a}} = \prod_{i=1}^{r} [\pi\mu_i\gamma(\mathbf{e}_i^2 b^2/2)]^{\boldsymbol{\omega}_i^\vee\cdot\boldsymbol{a}/b} \prod_{\boldsymbol{\alpha}>0} \Gamma(1-a_{\boldsymbol{\alpha}^\vee}/b)\,\Gamma(1-a_{\boldsymbol{\alpha}}b) \tag{19}$$

here $a_{\boldsymbol{\alpha}} = \boldsymbol{a}\cdot\boldsymbol{\alpha}$, $a_{\boldsymbol{\alpha}^\vee} = \boldsymbol{a}\cdot\boldsymbol{\alpha}^\vee$ and vectors $\boldsymbol{\omega}_i^\vee$ are the co-weights of G, satisfying the condition $\boldsymbol{\omega}_i^\vee\cdot\mathbf{e}_j = \delta_{ij}$.

The reflection relation Eq. (8) can be written in more symmetric form as:

$$A_{\boldsymbol{a}} V_{\boldsymbol{a}}(x) = A_{\hat{s}\boldsymbol{a}} V_{\hat{s}\boldsymbol{a}}(x), \qquad \hat{s}\in\mathscr{W} \tag{20}$$

In following we will be interested in the values of functions $A_{\boldsymbol{a}}$ for imaginary $\boldsymbol{a} = i\mathbf{P}$. We denote as $V(\mathbf{P}, x) = V_{i\mathbf{P}}(x)$ and $A(\mathbf{P}) = A_{i\mathbf{P}}$. Using these objects we can construct the combination which is invariant under the Weyl reflections:

$$\Psi_{\mathbf{P}} = \sum_{\hat{s}\in\mathscr{W}} A(\hat{s}\mathbf{P})\, V(\hat{s}\mathbf{P}) \tag{21}$$

From the above expression, we can find

$$\begin{aligned} \frac{A(\hat{s}_i\mathbf{P})}{A(\mathbf{P})} &= S_L(\mathbf{e}_i, \mathbf{P}) \\ &= [\pi\mu_i\gamma(\mathbf{e}_i^2 b^2/2)]^{-i\mathbf{P}\cdot\mathbf{e}_i^\vee/b} \frac{\Gamma(1+i\mathbf{P}\cdot\mathbf{e}_i b)\,\Gamma(1+i\mathbf{P}\cdot\mathbf{e}_i^\vee/b)}{\Gamma(1-i\mathbf{P}\cdot\mathbf{e}_i b)\,\Gamma(1-i\mathbf{P}\cdot\mathbf{e}_i^\vee/b)} \end{aligned} \tag{22}$$

From Eq. (22), we see that the functional form of S_L is independent of the algebra under consideration. Considering the simplest case (A_1 Toda), we identify that S_L is the reflection amplitude of LFT $R(iP) = S_L(\sqrt{2}\,P)$ where

$$S_L(P) = (\pi\mu\gamma(b^2))^{-iP/b} \frac{\Gamma(1+iPb)\,\Gamma(1+iP/b)}{\Gamma(1-iPb)\,\Gamma(1-iP/b)} \tag{23}$$

2.2.2. Super-LFT

Due to the superconformal symmetry, the primary fields of the super-LFT can be devided into two sectors. The (NS) primary fields are given by

$$V_\alpha(z,\bar z,\theta,\bar\theta)=\phi_\alpha(z,\bar z)+\theta\psi_\alpha(z,\bar z)+\bar\theta\bar\psi_\alpha(z,\bar z)-\theta\bar\theta\tilde\phi_\alpha(z,\bar z)$$

with dimensions

$$\Delta_\alpha=\tfrac{1}{2}\alpha(Q-\alpha)$$

and the (R) fields by

$$\sigma^{(\varepsilon)}V_\alpha$$

where $\sigma^{(\varepsilon)}$ is the "twist field" with dimension 1/16 so that the dimension of the (R) fields are

$$\Delta_\alpha=\tfrac{1}{16}+\tfrac{1}{2}\alpha(Q-\alpha)$$

The reflection amplitudes of the super-LFT defined from the structure constants have been derived from the structure constants in refs. 20 and 21. The reflection amplitudes for the (NS) fields are

$$S_{\mathrm{NS}}(P)=-\left(\frac{\pi\mu}{2}\gamma\left(\frac{1+b^2}{2}\right)\right)^{-2iP/b}\frac{\Gamma(1+iPb)\,\Gamma(1+iP/b)}{\Gamma(1-iPb)\,\Gamma(1-iP/b)} \tag{24}$$

and for the (R) fields

$$S_{\mathrm{R}}(P)=\left(\frac{\pi\mu}{2}\gamma\left(\frac{1+b^2}{2}\right)\right)^{-2iP/b}\frac{\Gamma(1/2+iPb)\,\Gamma(1/2+iP/b)}{\Gamma(1/2-iPb)\,\Gamma(1/2-iP/b)} \tag{25}$$

3. REFLECTIONS OF ZERO-MODES

In this section we introduce the wave functional interpretation of the primary fields of the LFT using the zero-mode following ref. 6 and show how to generalize it to NATTs by considering the wave functionals $\Psi[\boldsymbol{\varphi}(x)]$ whose asymptotic behaviours are described by the wave functions of the zero-modes in higher dimensions.

3.1. LFT

Consider LFT on a cylinder of circumference 2π with the cartesian coordinates x_1, x_2 where x_2 along the cylinder is defined as the imaginary time and $x_1 \sim x_1 + 2\pi$ is the space coordinate. The Hamiltonian acting in the space of states $\mathscr{A}$ of LFT

$$H = -\frac{c_L}{12} + L_0 + \bar{L}_0 \tag{26}$$

generates translations along the time x_2.

The conformal structure can be formulated in terms of the "zero-mode" of the Liouville field $\phi(x)$ defined by

$$\phi_0 = \int_0^{2\pi} \phi(x) \frac{dx_1}{2\pi} \tag{27}$$

As $\phi_0 \to -\infty$ in the configuration space, one can neglect the exponential interaction term in the LFT action so that one can expand $\phi(x)$ as a free massless field ($z = x_1 + ix_2$)

$$\phi(x) = \phi_0 - \mathscr{P}(z - \bar{z}) + \sum_{n \neq 0} \left(\frac{ia_n}{n} e^{inz} + \frac{i\bar{a}_n}{n} e^{in\bar{z}} \right) \tag{28}$$

where the momentum conjugate to the zero-mode ϕ_0 and oscillators satisfy

$$\mathscr{P} = -\frac{i}{2} \frac{\partial}{\partial \phi_0}, \qquad [a_m, a_n] = \frac{m}{2} \delta_{m+n}, \qquad [\bar{a}_m, \bar{a}_n] = \frac{m}{2} \delta_{m+n} \tag{29}$$

The Virasoro generators can be written in terms of these modes. The space of states is now represented as

$$\mathscr{A}_0 = \mathscr{L}_2(-\infty < \phi_0 < \infty) \otimes \mathscr{F} \tag{30}$$

where $\mathscr{L}_2$ is the two-dimensional phase space spanned by ϕ_0 and its conjugate momentum $\mathscr{P}$ and $\mathscr{F}$ is the Fock space of the oscillators.

Any state $s \in \mathscr{A}$ can be represented by a wave functional $\Psi_s[\phi(x_1)]$ in the $\phi_0 \to -\infty$ asymptotic limit. In particular, the wave functional for the primary state v_P corresponds to

$$\Psi_{v_P}[\phi(x_1)] = (e^{iP\phi_0} + S(P)\, e^{-iP\phi_0})\, |0\rangle \qquad \text{as} \quad \phi_0 \to -\infty \tag{31}$$

where $S(P)$ is the reflection coefficient of the asymptotic wave functional. One can check that the wave functional of asymptotic form Eq. (31) has correct conformal dimension by acting L_0. The coefficient $S(P)$ should be the reflection amplitude $S_L(P)$ introduced earlier since the wave functional $\Psi_{v_{-P}}$ for the primary state v_{-P} is $S_L(-P)\,\Psi_{v_P}$ along with

$$S_L(P)\,S_L(-P)=1 \tag{32}$$

In this framework, one can check the validity of the reflection amplitude by taking semiclassical limit $b\to 0$ and using duality. Since P is of the order of $\mathcal{O}(b)$, one can neglect the oscillators and keep only the zero-mode ϕ_0 so that the Hamiltonian is approximated as

$$H_0=-\tfrac{1}{12}-\mathscr{P}^2+2\pi\mu e^{\sqrt{2}\,b\phi_0} \tag{33}$$

The exact wave function of ϕ_0 for the Hamiltonian is well-known whose asymptotic form as $\phi_0\to-\infty$ is given by Eq. (31) with

$$S(P)=-\left(\frac{\pi\mu}{b^2}\right)^{-iP/b}\frac{\Gamma(1+iP/b)}{\Gamma(1-iP/b)} \tag{34}$$

It is straightforward to check this result is consistent with the non-perturbative reflection amplitude Eq. (23) perturbatively.

3.2. NATTs

The zero-modes of the fields $\boldsymbol{\varphi}(x)$ are defined as:

$$\boldsymbol{\varphi}_0=\int_0^{2\pi}\boldsymbol{\varphi}(x)\,\frac{dx_1}{2\pi} \tag{35}$$

Here we consider the NATT on an infinite plane cylinder of circumference 2π with coordinate x_2 along the cylinder playing the role of imaginary time. In the asymptotic region where the potential terms in the NATT action become negligible ($\mathbf{e}_i\cdot\boldsymbol{\varphi}_0\to-\infty$ for all i), the fields can be expanded in terms of free field operators $\mathbf{a}_n$

$$\boldsymbol{\varphi}(x)=\boldsymbol{\varphi}_0-\mathscr{P}(z-\bar{z})+\sum_{n\neq 0}\left(\frac{i\mathbf{a}_n}{n}\,e^{inz}+\frac{i\bar{\mathbf{a}}_n}{n}\,e^{-in\bar{z}}\right) \tag{36}$$

where $\mathscr{P}=-i\nabla_{\boldsymbol{\varphi}_0}$ is the conjugate momentum of $\boldsymbol{\varphi}_0$. In this region any state of the NATT can be decomposed into a direct product of two parts, namely, a wave function of the zero-modes and a state in Fock space generated by the operators $\mathbf{a}_n$. In particular, the wave functional corresponding to the primary state Eq. (21) can be expressed as a direct product of a wave function of the zero-modes $\boldsymbol{\varphi}_0$ and Fock vacuum:

$$\Psi_{\mathbf{P}}[\boldsymbol{\varphi}(x)]\sim\Psi_{\mathbf{P}}(\boldsymbol{\varphi}_0)\otimes|0\rangle \tag{37}$$

where the wave function $\Psi_{\mathbf{P}}(\boldsymbol{\varphi}_0)$ in this asymptotic region is a superposition of plane waves with momenta $\hat{s}\mathbf{P}$.

The reflection amplitudes of the NATT defined in the previous section can be interpreted as those for the wave function of the zero-modes in the presence of potential walls. This can be understood most clearly in the semiclassical limit $b \to 0$ where one can neglect the operators $\mathbf{a}_n$ in Eq. (36) even for significant values of the parameters μ_i. The full quantum effect can be implemented simply by introducing the exact reflection amplitudes which take into account also non-zero-mode contributions.[6] The resulting Schrödinger equation is given by

$$\left[-\frac{r}{12} - \nabla^2_{\boldsymbol{\varphi}_0} + \sum_{i=1}^{r} 2\pi\mu_i e^{b\mathbf{e}_i \cdot \boldsymbol{\varphi}_0}\right] \Psi_{\mathbf{P}}(\boldsymbol{\varphi}_0) = E_0 \Psi_{\mathbf{P}}(\boldsymbol{\varphi}_0) \tag{38}$$

with the ground state energy

$$E_0 = -\frac{r}{12} + \mathbf{P}^2 \tag{39}$$

Here the momentum $\mathbf{P}$ is any continuous real vector. The effective central charge can be obtained from Eq. (39) where $\mathbf{P}^2$ takes the minimal possible value for the perturbed theory. Since only asymptotic form of the wave function matters, we derive the reflection amplitudes of the ATFTs in the way that we need only the LFT result.

In the $\mu_i \to 0$ limit which will be of our interest, the potential vanishes almost everywhere except for the values of $\boldsymbol{\varphi}_0$ where some of exponential terms in the potential become large enough to overcome the small value of μ_i. In this case, each exponential term $e^{b\mathbf{e}_i \cdot \boldsymbol{\varphi}_0}$ in the interaction represents a wall with $\mathbf{e}_i$ being its normal vector. If we consider the behaviour of a wave function near a wall normal to $\mathbf{e}_i$ where the effect of other interaction terms becomes negligible, the problem becomes equivalent to the LFT in the $\mathbf{e}_i$ direction. The potential becomes flat in the $(r-1)$-dimensional orthogonal directions. The asymptotic form of the energy eigenfunction is then given by the product of that of Liouville wave function and $(r-1)$-dimensional plane wave,

$$\begin{aligned} \Psi &\sim [e^{iP_i\varphi_{0i}} + S_L(\mathbf{e}_i, vP)\, e^{-iP_i\varphi_{0i}}]\, e^{i\mathbf{P}_\perp \cdot \boldsymbol{\varphi}_0} \\ &\sim e^{i\mathbf{P}\cdot\boldsymbol{\varphi}_0} + S_L(\mathbf{e}_i, \mathbf{P})\, e^{i\hat{s}_i\mathbf{P}\cdot\boldsymbol{\varphi}_0} \end{aligned} \tag{40}$$

where $\hat{s}_i$ denotes the Weyl reflection by the simple root $\mathbf{e}_i$ and P_i the component of $\mathbf{P}$ along $\mathbf{e}_i$ direction. $S_L(\mathbf{e}_i, \mathbf{P})$ is defined in Eq. (22). Since the wave function interpretation makes sense only in the semiclassical limit, it is the $b \to 0$ limit of Eq. (23) which can be obtained from the solution of the Schrödinger equation for the LFT.

We can see from Eq. (40) that the momentum of the reflected wave by the ith wall is given by the Weyl reflection $\hat{s}_i$ acting on the incoming momentum. If we consider the reflections from all the potential walls, the wave function in the asymptotic region is a superposition of the plane waves reflected by potential walls in different ways. The momenta of these waves form the orbit of the Weyl group $\mathscr{W}$ of the Lie algebra G;

$$\Psi_{\mathbf{P}}(\boldsymbol{\varphi}_0) \simeq \sum_{\hat{s} \in \mathscr{W}} A(\hat{s}\mathbf{P})\, e^{i\hat{s}\mathbf{P} \cdot \boldsymbol{\varphi}_0} \tag{41}$$

This is indeed the wave function representation of the primary field (21) in the asymptotic region. It follows from Eq. (40) that the amplitudes $A(\mathbf{P})$ satisfy the relations

$$\frac{A(\hat{s}_i \mathbf{P})}{A(\mathbf{P})} = S_L(\mathbf{e}_i, \mathbf{P}) \tag{42}$$

which is the same as Eq. (23). Equation (42) is solved by

$$A(\mathbf{P}) = \prod_{i=1}^{r} [\pi \mu_i \gamma(\mathbf{e}_i^2 b^2/2)]^{i\boldsymbol{\omega}_i^\vee \cdot \mathbf{P}/b} \prod_{\boldsymbol{\alpha}>0} \Gamma(1 - iP_{\boldsymbol{\alpha}} b)\, \Gamma(1 - iP_{\boldsymbol{\alpha}^\vee}/b) \tag{43}$$

and we recover the result (19) calculated in the previous section.

3.3. Super-LFT

The super-LFT is a super-CFT which satisfies the usual super-Virasoro algebra. The space of states for the super-LFT can be expressed by

$$\mathscr{A}_0 = \mathscr{L}_2(-\infty < \phi_0 < \infty, \psi_0) \otimes \mathscr{F} \tag{44}$$

where the fermionic zero-mode appears only for the (R) sector and $\mathscr{F}$ is the Fock space of bosonic and fermionic oscillators. The appearance of bosonic and fermionic zero-modes in Eq. (44) is well-known from the super-CFT results. In the (NS) sector, there is no fermionic zero-mode since the fermion field satisfies the anti-periodic boundary condition while it appears in the (R) sector with periodic one. The zero-modes appear in the super-Virasoro generator L_0 and S_0 of the (R) sector in such a way that L_0 contains the square of the conjugate momentum and S_0 acts non-trivially only on the twist field.

The primary state v_P can be also expressed by a wave functional $\Psi_{v_P}[\phi(x_1)]$ whose asymptotic form is given similarly as Eq. (40). The amplitude $S(P)$ is either $S_{\rm NS}(P)$ or $S_{\rm R}(P)$ depending on the sector so that the wave functional $\Psi_{v_{-P}}$ is given by $S(-P)\,\Psi_{v_P}$. One can also check the validity of this expression by taking the classical limit of $b \to 0$. Since P is small of order of $\mathcal{O}(b)$, one can neglect the oscillator part in Eq. (44) and study only the dynamics of zero-modes. In the (NS) sector, only bosonic zero-mode appears so that the Hamiltonian becomes

$$H_0^{\rm NS} = -\frac{1}{8} - \left(\frac{\partial}{\partial \phi_0}\right)^2 + \pi^2\mu^2 b^2 e^{2b\phi_0}$$

which is essentially the same as that of the LFT, hence the reflection amplitude becomes

$$S_{\rm NS}(P) = -\left(\frac{\pi\mu}{2}\right)^{-(2i/b)\,P} \frac{\Gamma(1+iP/b)}{\Gamma(1-iP/b)}$$

On the other hand, in the (R) sector, additional fermionic zero-mode is introduced in the hamiltonian by[22]

$$H_0^{\rm R} = -\left(\frac{\partial}{\partial \phi_0}\right)^2 + \pi^2\mu^2 b^2 e^{2b\phi_0} + 2\pi i\mu b^2 \psi_0 \bar{\psi}_0 e^{b\phi_0}$$

Since the fermionic zero-mode satisfies

$$\{\psi_0, \bar{\psi}_0\} = 0, \qquad \psi_0^2 = \bar{\psi}_0^2 = \tfrac{1}{2}$$

we can represent it by

$$\psi_0 = \frac{1}{\sqrt{2}}\,\sigma_1, \qquad \bar{\psi}_0 = \frac{1}{\sqrt{2}}\,\sigma_2, \qquad \psi_0\bar{\psi}_0 = \frac{i}{2}\,\sigma_3$$

and the Hamiltonian becomes

$$H_0^{\rm R} = \mathscr{P}^2 + \pi^2\mu^2 b^2 e^{2b\phi_0} - \pi\mu b^2 e^{b\phi_0}\sigma_3 = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}$$

The solution of H_+ can be obtained as

$$\Psi_+(\phi_0) = \begin{pmatrix} \sqrt{x}\,[K_{1/2-iP/b}(x) + K_{1/2+iP/b}(x)] \\ 0 \end{pmatrix}, \qquad x = \pi\mu e^{b\phi_0}$$

where $K_\nu(x)$ is the modified Bessel function. By taking the asymptotic limit $\phi_0 \to -\infty$, one can find the non-vanishing component is given by

$$\Psi \sim e^{iP\phi_0} + S_{\mathbf{R}}(P)\, e^{-iP\phi_0}$$

with

$$S_{\mathbf{R}}(P) = \left(\frac{\pi\mu}{2}\right)^{-(2i/b)\,P} \frac{\Gamma(1/2 + iP/b)}{\Gamma(1/2 - iP/b)}$$

These are consistent with the exact result Eq. (25) in the $b \to 0$ limit.

4. QUANTIZATION CONDITIONS AND SCALING FUNCTIONS

In this section we derive the scaling functions for the various Toda-type models on a cylinder with circumference R. In the deep UV region $R \to 0$, the wave functional interpretation introduced in the previous section is used to obtain the quantization condition for the zero-mode momenta and the vacuum energies.

4.1. ShG Model

We start by reviewing the analysis of ref. 6 for the ShG model or A_1 ATFT defined first on a circle of circumference R with periodic boundary condition. By rescaling the size to 2π, one can write the action as

$$\mathcal{A}_{\text{ShG}} = \int dx_2 \int_0^{2\pi} dx_1 \left[\frac{1}{4\pi} (\partial_a \phi)^2 + \mu \left(\frac{R}{2\pi}\right)^{2+2b^2} (e^{2b\phi} + e^{-2b\phi}) \right] \tag{45}$$

where $\mu \sim [\text{mass}]^{2+2b^2}$ is the dimensional coupling constant with b the coupling constant.

We are interested in the ground state energy $E(R)$ or, more conveniently, the finite-size effective central charge

$$c_{\text{eff}}(R) = -\frac{6R}{\pi} E(R) \tag{46}$$

in the ultraviolet limit $R \to 0$. Since we are interested in the ground-state energy, only the zero-mode contribution counts. So the corresponding effective central charge at $R \to 0$ is determined mainly by P

$$c_{\text{eff}}(R) = 1 - 24P^2 + \mathcal{O}(R) \tag{47}$$

up to power corrections in R.

For the ground state energy, one can consider only the zero-mode dynamics where the wave functional of ϕ_0 is confined in the potential well due to the ShG interaction term. The ShG potential introduces a quantization condition for the momentum P which depends on the finite size R. As $R \to 0$, in particular, the wave functional is confined in the potential well where the potential vanishes in the most of the region and becomes non-trivial only at $2b\phi_0 \sim \pm \ln \mu(R/2\pi)^{2+2b^2}$ near the left and right edges. Near these edges of the potential well, the potential becomes that of the LFT and the wave functional will be reflected with the reflection amplitude of the LFT introduced earlier. Therefore, the quantization condition is given by

$$(R/2\pi)^{-4\sqrt{2}\,iPQ}\, S_L^2(P) = 1 \tag{48}$$

In terms of the reflection phase $\delta_L(P)$ defined by

$$S_L(P) = -e^{i\delta_L(P)} \tag{49}$$

the ground state momentum is qunatized as

$$\delta_L(P) = \pi + 2\sqrt{2}\, PQ \ln \frac{R}{2\pi} \tag{50}$$

Thus determined quantized momentum will give the scaling function $c_{\text{eff}}(R)$ in the UV region by Eq. (47). To see this explicitly, one can expand the reflection phase in the odd powers of P,

$$\delta_L(P) = \delta_1(b)\, P + \delta_3(b)\, P^3 + \delta_5(b)\, P^5 + \cdots \tag{51}$$

where the coefficients can be obtained from the reflection amplitude Eq. (23) as follows:

$$\delta_1(b) = \frac{2}{b} \ln b^2 - 2Q \ln \left[\frac{\Gamma(1/(2+2b^2))\, \Gamma(1+b^2/(2+2b^2))}{4\sqrt{\pi}} + \gamma_{\text{E}} \right]$$

$$\delta_3(b) = \frac{2}{3} \zeta(3)(b^3 + b^{-3})$$

$$\delta_5(b) = -\frac{2}{5} \zeta(5)(b^5 + b^{-5})$$

with Euler constant γ_{E}. Now solving Eq. (51) iteratively, we get

$$c_{\text{eff}}(R) = 1 + \frac{c_1}{l^2} + \frac{c_2}{l^5} + \frac{c_3}{l^7} + \cdots \tag{52}$$

where

$$l = \delta_1(b) - 2\sqrt{2}\, Q \ln(R/2\pi) \tag{53}$$

$$c_1 = -6\pi^2, \qquad c_2 = 12\pi^4 \delta_3(b), \qquad c_3 = 12\pi^6 \delta_5(b) \tag{54}$$

The Gamma functions appear in δ_1 due to the relation between the mass of the physical particle and the coupling constant μ in the action[18]

$$-\frac{\pi\mu}{\gamma(-b^2)} = \left[\frac{m}{4\sqrt{\pi}}\,\Gamma\left(\frac{1}{2+2b^2}\right)\Gamma\left(1+\frac{b^2}{2+2b^2}\right)\right]^{2+2b^2} \tag{55}$$

4.2. ATFTs

The action of the ATFT is given by

$$\mathcal{A} = \int d^2x \left[\frac{1}{8\pi}(\partial_\mu \boldsymbol{\varphi})^2 + \sum_{i=1}^{r} \mu_i e^{b\mathbf{e}_i \cdot \boldsymbol{\varphi}} + \mu_0 e^{b\mathbf{e}_0 \cdot \boldsymbol{\varphi}}\right] \tag{56}$$

The additional potential term in the ATFT Lagrangian corresponding to the zeroth root $\mathbf{e}_0$ introduces new potential wall in that direction (see Fig. 1 as a simplest example, the A_2 ATFT). With this addition, the region of $\boldsymbol{\varphi}_0$ made of the non-affine Toda potential walls (Weyl chamber) is now closed and the momentum of the wave function should be quantized depending

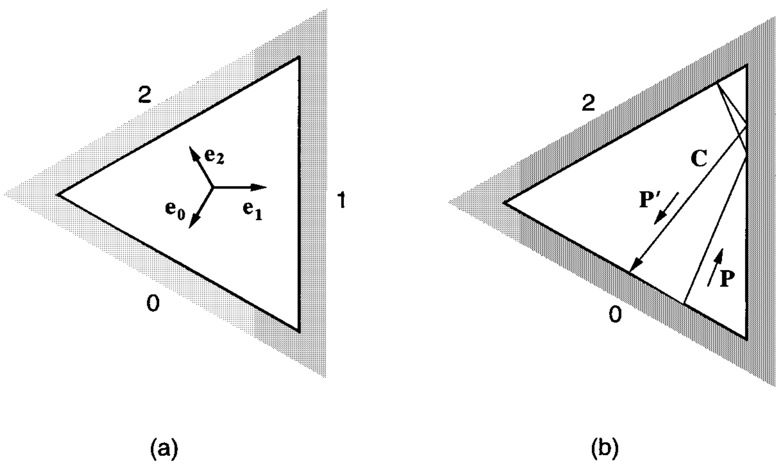

Fig. 1. (a) Potential walls in A_2 affine Toda theory. (b) A wave with momentum **P** near the zeroth wall comes back to the same wall with momentum **P′** after a series of reflections.

on the size of the enclosed region. This quantized momentum defines the scaling function c_{eff} in the UV region by Eq. (39).

The quantization condition can be derived as follows. For simplicity let us first consider simply laced cases. Also, we assume for the moment that the circumference of the cylinder is 2π. Consider the path C of a wave which starts with momentum $\mathbf{P}$ and comes back (after a series of reflections by other walls) to the zeroth potential wall with momentum $\mathbf{P}'$. It will then be reflected by the zeroth wall. Figure 1(b) illustrates a multiple reflection in the two-dimensional potential. To satisfy the self-consistency condition, the momentum $\mathbf{P}'$ after the last reflection by the zeroth wall should be equal to the incoming momentum $\mathbf{P}$ so that $s_0\mathbf{P}'=\mathbf{P}$. Furthermore, since the zeroth wall is again Liouville-type, the momenta $\mathbf{P}'=s_0\mathbf{P}$ of the *incident* wave and $\mathbf{P}$ of the *reflected* wave should satisfy Eq. (42) which leads to

$$\frac{A(s_0\mathbf{P})}{A(\mathbf{P})}=S_L(\mathbf{e}_0,\mathbf{P}) \tag{57}$$

On the other hand, since s_0 is given by a product of the Weyl reflections corresponding to simple roots, each representing the reflection experienced by the wave along the path C, the left hand side of Eq. (57) can be obtained from Eq. (43). Therefore, Eq. (57) gives a nontrivial quantization condition for the momentum $\mathbf{P}$. This condition can be generalized using the same arguments for other potential walls instead of the zeroth one. Then we obtain

$$\frac{A(s_0 s\mathbf{P})}{A(s\mathbf{P})}=S_L(\mathbf{e}_0,s\mathbf{P}) \tag{58}$$

where s is an arbitrary Weyl group element.

Using Eq. (18) we can write (58)

$$(\pi\mu\gamma(b^2))^{i\mathbf{P}\cdot s\mathbf{v}/b}\left[\prod_{\boldsymbol{\alpha}>0}\frac{G(s\mathbf{P}\cdot s_0\boldsymbol{\alpha})}{G(s\mathbf{P}\cdot\boldsymbol{\alpha})}\right]\frac{G(s\mathbf{P}\cdot\mathbf{e}_0)}{G(-s\mathbf{P}\cdot\mathbf{e}_0)}=1 \tag{59}$$

where

$$\mathbf{v}=s_0\boldsymbol{\rho}-\boldsymbol{\rho}+\mathbf{e}_0=-(\mathbf{e}_0\cdot\boldsymbol{\rho})\,\mathbf{e}_0+\mathbf{e}_0=h\mathbf{e}_0$$

and we define a function

$$G(P)=\Gamma(1-iP/b)\,\Gamma(1-iPb)$$

The Γ-function factors in Eq. (59) can be further simplified. First, consider the action of s_0 on a positive root $\boldsymbol{\alpha}$: $s_0\boldsymbol{\alpha} = \boldsymbol{\alpha} - \mathbf{e}_0(\mathbf{e}_0 \cdot \boldsymbol{\alpha})$, which is either $\boldsymbol{\alpha}$ or $\boldsymbol{\alpha} + \mathbf{e}_0$ if $\boldsymbol{\alpha} \neq -\mathbf{e}_0$ since $-\mathbf{e}_0$ is the maximal root. In the first case, the factor $G(s\mathbf{P} \cdot s_0\boldsymbol{\alpha})$ in Eq. (59) is cancelled out by the same factor in the denominator, while, in the second case, there is no cancellation since $\boldsymbol{\alpha} + \mathbf{e}_0$ is a negative root. Finally, $s_0\mathbf{e}_0 = -\mathbf{e}_0$ and the corresponding factor $G(s\mathbf{P} \cdot \mathbf{e}_0)$ appears twice in Eq. (58). Using the property $\mathbf{e}_0 \cdot \boldsymbol{\alpha} = 0$ or 1 for $\boldsymbol{\alpha} > 0$ $(\boldsymbol{\alpha} \neq -\mathbf{e}_0)$ and $\mathbf{e}_0 \cdot \mathbf{e}_0 = 2$, we can simplify Eq. (59) as

$$(\pi\mu\gamma(b^2))^{ih\mathbf{P} \cdot s\mathbf{e}_0/b} \prod_{\boldsymbol{\alpha}>0} \left[\frac{G(-\mathbf{P} \cdot s\boldsymbol{\alpha})}{G(\mathbf{P} \cdot s\boldsymbol{\alpha})} \right]^{-s\boldsymbol{\alpha} \cdot s\mathbf{e}_0} = 1 \tag{60}$$

For the non-simply laced ATFTs, we obtain the condition for $\mathbf{P}$ by inserting Eqs. (43) and (22) into Eq. (58). After some transformations as above, it can be written in the form:

$$\left[\prod_{i=0}^{r} (\pi\mu_i\gamma(\mathbf{e}_i^2 b^2/2))^{n_i} \right]^{i\mathbf{P} \cdot s\mathbf{e}_0^\vee/b} \prod_{\boldsymbol{\alpha}>0} \left[\frac{\mathcal{G}(\boldsymbol{\alpha}, \mathbf{P})}{\mathcal{G}(\boldsymbol{\alpha}, -\mathbf{P})} \right]^{\boldsymbol{\alpha} \cdot s\mathbf{e}_0^\vee} = 1 \tag{61}$$

where we define

$$\mathcal{G}(\boldsymbol{\alpha}, \mathbf{P}) = \Gamma(1 - iP_{\boldsymbol{\alpha}}b)\,\Gamma(1 - iP_{\boldsymbol{\alpha}^\vee}/b)$$

Now we consider the system defined on a cylinder with the circumference R. When we scale back the size from R to 2π, the parameters μ_i in the action (56) rescales as

$$\mu_i \to \mu_i \left(\frac{R}{2\pi} \right)^{2 + b^2\mathbf{e}_i^2} \tag{62}$$

Then, Eq. (61) for the lowest energy state reduces to

$$L\mathbf{P} = 2\pi\boldsymbol{\rho} - \sum_{\boldsymbol{\alpha}>0} \boldsymbol{\alpha}\delta(\boldsymbol{\alpha}, \mathbf{P}) \tag{63}$$

where

$$L = -\frac{2}{b}(h + b^2 h^\vee) \ln \frac{R}{2\pi} - \frac{1}{b} \ln \left[\prod_{i=0}^{r} (\pi\mu_i\gamma(\mathbf{e}_i^2 b^2/2))^{n_i} \right] \tag{64}$$

and

$$\delta(\boldsymbol{\alpha}, \mathbf{P}) = -i \log \frac{\Gamma(1 + iP_{\boldsymbol{\alpha}}b)\,\Gamma(1 + iP_{\boldsymbol{\alpha}^\vee}/b)}{\Gamma(1 - iP_{\boldsymbol{\alpha}}b)\,\Gamma(1 - iP_{\boldsymbol{\alpha}^\vee}/b)} \tag{65}$$

This is the quantization condition for the momentum $\mathbf{P}$ in the UV region $R \to 0$.

The ground state energy with the circumference R is given by

$$E(R) = -\frac{\pi c_{\text{eff}}}{6R} \qquad \text{with} \qquad c_{\text{eff}} = r - 12\mathbf{P}^2 \tag{66}$$

where $\mathbf{P}$ satisfies Eq. (63).

In the UV region we can solve Eq. (63) perturbatively by expanding $\delta(\boldsymbol{\alpha}, \mathbf{P})$ in powers of $P_{\boldsymbol{\alpha}}$,

$$\delta(\boldsymbol{\alpha}, \mathbf{P}) = \delta_1(\boldsymbol{\alpha}, b)\, P_{\boldsymbol{\alpha}} + \delta_3(\boldsymbol{\alpha}, b)\, P_{\boldsymbol{\alpha}}^3 + \delta_5(\boldsymbol{\alpha}, b)\, P_{\boldsymbol{\alpha}}^5 \cdots \tag{67}$$

where the coefficients $\delta_1(\boldsymbol{\alpha}, b)$ and $\delta_s(\boldsymbol{\alpha}, b)$, $s = 3, 5$ are:

$$\delta_1(\boldsymbol{\alpha}, b) = -2\gamma_{\mathrm{E}}\left(b + \frac{2}{\boldsymbol{\alpha}^2 b}\right), \qquad \delta_s(\boldsymbol{\alpha}, b) = (-)^{(s-3)/2} \cdot \frac{2}{s}\, \zeta(s) \left(b^s + \left(\frac{2}{\boldsymbol{\alpha}^2 b}\right)^s\right) \tag{68}$$

Using the relations: $\sum_{\boldsymbol{\alpha}>0} (\boldsymbol{\alpha})^a\, (\boldsymbol{\alpha})^b = h^{\vee} \delta^{ab}$, and $\sum_{\boldsymbol{\alpha}>0} (\boldsymbol{\alpha})^a\, (\boldsymbol{\alpha}^{\vee})^b = h\delta^{ab}$, we obtain that:

$$l\mathbf{P} = 2\pi\boldsymbol{\rho} - \sum_{\boldsymbol{\alpha}>0} \delta_3(\boldsymbol{\alpha}, b)\, \boldsymbol{\alpha} P_{\boldsymbol{\alpha}}^3 - \sum_{\boldsymbol{\alpha}>0} \delta_5(\boldsymbol{\alpha}, b)\, \boldsymbol{\alpha} P_{\boldsymbol{\alpha}}^5 - \cdots$$

with

$$l = L - 2\gamma_{\mathrm{E}}(bh^{\vee} + h/b) \equiv L - L_0 \tag{69}$$

The above equation can be solved iteratively in powers of $1/l$. Inserting the solution into Eq. (66), we find:

$$c_{\text{eff}} = r - r(h+1)\, h^{\vee} \left(\frac{2\pi}{l}\right)^2 + \frac{8}{\pi}\, \zeta(3) [C_4(G^{\vee})\, b^3 + C_4(G)/b^3] \left(\frac{2\pi}{l}\right)^5 - \frac{24}{5\pi}\, \zeta(5) [C_6(G^{\vee})\, b^5 + C_6(G)/b^5] \left(\frac{2\pi}{l}\right)^7 + \mathcal{O}(l^{-8}) \tag{70}$$

where the coefficients $C(G)$ are defined as:

$$C_4(G) = \sum_{\boldsymbol{\alpha}>0} \rho_{\boldsymbol{\alpha}} \rho_{\boldsymbol{\alpha}^{\vee}}^3, \qquad C_4(G^{\vee}) = \sum_{\boldsymbol{\alpha}>0} \rho_{\boldsymbol{\alpha}}^4$$

$$C_6(G) = \sum_{\boldsymbol{\alpha}>0} \rho_{\boldsymbol{\alpha}} \rho_{\boldsymbol{\alpha}^{\vee}}^5, \qquad C_6(G^{\vee}) = \sum_{\boldsymbol{\alpha}>0} \rho_{\boldsymbol{\alpha}}^6$$

For simply laced algebras, these coefficients have the values:

$$\begin{aligned} C_4(A_{n-1}^{(1)}) &= \tfrac{1}{60}n^2(n^2-1)(2n^2-3) \\ C_6(A_{n-1}^{(1)}) &= \tfrac{1}{168}n^2(n^2-1)(n^2-2)(3n^2-5) \\ C_4(D_n^{(1)}) &= \tfrac{1}{30}(16n^3-45n^2+27n+8)\, n(n-1)(2n-1) \\ C_6(D_n^{(1)}) &= \tfrac{1}{42}(48n^5-213n^4+262n^3+6n^2-101n-32)\, n(n-1)(2n-1) \end{aligned} \tag{71}$$

For the non-simply laced algebras $B_n^{(1)}$ and $C_n^{(1)}$, we can express the results through these values. Namely, we find:

$$\begin{aligned} C_i(B_n^{(1)}) = \tfrac{1}{2}C_i(A_{2n-1}^{(1)}), \qquad C_i(B_n^{(1)\vee}) = C_i(D_{n+1/2}^{(1)}) \\ C_i(C_n^{(1)}) = C_i(D_{n+1}^{(1)}), \qquad C_i(C_n^{(1)\vee}) = C_i(D_{-n}^{(1)}), \qquad (i=4,6) \end{aligned} \tag{72}$$

For exceptional algebras $G_2^{(1)}$ and $F_4^{(1)}$, we obtain:

$$\begin{aligned} C_4(G_2^{(1)}) = \tfrac{1}{3}C_4(D_4^{(1)}) = 392, \qquad C_4(G_2^{(1)\vee}) = \tfrac{980}{9} \\ C_6(G_2^{(1)}) = \tfrac{1}{3}C_6(D_4^{(1)}) = 7386, \qquad C_6(G_2^{(1)\vee}) = \tfrac{199516}{243} \\ C_4(F_4^{(1)}) = \tfrac{1}{2}C_4(E_6^{(1)}) = 27378, \qquad C_4(F_4^{(1)\vee}) = \tfrac{22815}{2} \\ C_6(F_4^{(1)}) = \tfrac{1}{2}C_6(E_6^{(1)}) = 2203578, \qquad C_6(F_4^{(1)\vee}) = \tfrac{4052763}{8} \end{aligned} \tag{73}$$

We note that above equations relating coefficients $C_i(G)$ for different Lie algebras follow from the similar exact relations between the ground state energies $e(G)$ of quantum affine Toda chains associated with these Lie algebras. These exact relations are valid if the parameters μ, μ' for non-simply laced Lie algebras and corresponding parameter μ_{sl} for simply laced ones satisfy the condition: $\mu^{h-z}(2\mu'/\xi^2)^z = \mu_{sl}^h$, where $z = 2(h-h^\vee)/(2-\xi^2)$.

4.3. The BD Model

The BD model is an integrable field theory associated with $A_2^{(2)}$ affine Toda theory and can be regarded as an integrable perturbation of the LFT.[19] The action is given on a circle of circumference 2π with periodic boundary condition;

$$\mathcal{A}_{\mathrm{BD}} = \int dx_2 \int_0^{2\pi} dx_1 \left[\frac{1}{4\pi}(\partial_a\phi)^2 + \mu\left(\frac{R}{2\pi}\right)^{2+2b^2} e^{2b\phi} + \mu'\left(\frac{R}{2\pi}\right)^{2+b^2/2} e^{-b\phi} \right] \tag{74}$$

This model possesses asymmetrical exponential potential terms compared with the ShG model. In the UV limit, the exponential potential becomes negligibly small except in the region where ϕ_0 goes to $\pm\infty$. This means that the BD model is again effectively described by the LFT. It is the quantization condition that makes the difference from the ShG model, due to the asymmetry of the potential well in the left and right edges. The conjugate momentum P is now quantized by the condition

$$\left(\frac{R}{2\pi}\right)^{-4iP(Q+Q')} S_L(P)\, S'_L(P) = 1 \tag{75}$$

where $S'_L(P)$ is obtained by substituting $b \to b/2$ for $S_L(P)$ given in Eq. (23) and

$$Q = b + 1/b, \qquad Q' = b/2 + 2/b \tag{76}$$

Using the phase shifts defined as

$$S_L(P) = -e^{i\delta_L(P)}, \qquad S'_L(P) = -e^{i\delta'_L(P)}$$

the quantization condition becomes

$$\bar{\delta}(P) = \pi + 4\bar{Q}P \ln\frac{R}{2\pi} \tag{77}$$

where

$$\bar{\delta}(P) = \tfrac{1}{2}(\delta_L(P) + \delta'_L(P)), \qquad \bar{Q} = \tfrac{1}{2}(Q + Q')$$

The relation between P and R in Eq. (77) gives the scaling function c_{eff} as a continuous function of R, Eq. (70), with Q replaced by $\bar{Q}$ and δ's with $\bar{\delta}$'s defined by power series expansion of the phase shift in P

$$\begin{aligned}
\bar{\delta}(P) &= \bar{\delta}_1 P + \bar{\delta}_3 P^3 + \bar{\delta}_5 P^5 + \cdots \\
\bar{\delta}_1 &= \frac{6}{b}\ln\frac{b^2}{2} - 2(Q+Q')\left\{\ln\left[-\frac{\mu\pi\Gamma(1+b^2)}{\Gamma(-b^2)}\right]^{1/(6+3b^2)}\right. \\
&\qquad \left.\times\left[-\frac{2\mu'\pi\Gamma(1+b^2/4)}{\Gamma(-b^2/4)}\right]^{2/(6+3b^2)} + \gamma_{\mathrm{E}}\right\} \\
\bar{\delta}_3 &= 3\zeta(3)\left(b^3 + \frac{8}{b^3}\right) \\
\bar{\delta}_5 &= -\frac{33}{5}\zeta(5)\left(b^5 + \frac{32}{b^5}\right)
\end{aligned} \tag{78}$$

Then, the central charge is given by

$$c_{\text{eff}}(R) = 1 + \frac{c_1}{l^2} + \frac{c_2}{l^5} + \frac{c_3}{l^7} + \cdots \tag{79}$$

where

$$\begin{aligned} l &= \bar{\delta}_1(b) - 4Q \ln(R/2\pi) \\ c_1 &= -24\pi^2 \\ c_2 &= 48\pi^4\, \bar{\delta}_3(b) \\ c_3 &= 48\pi^6\, \bar{\delta}_5(b) \end{aligned} \tag{80}$$

4.4. The SShG Model

Now we consider an integrable model obtained as a perturbation of the super-LFT, the SShG model. By rescaling the size to 2π, one can express the action of the SShG model by

$$\begin{aligned} \mathcal{A}_{\text{SShG}} = \int dx_2 \int_0^{2\pi} dx_1 \Bigg[&\frac{1}{8\pi}(\partial_a \phi)^2 - \frac{1}{2\pi}(\bar{\psi}\, \partial \bar{\psi} + \psi\, \bar{\partial} \psi) \\ &+ 2i\mu b^2 \left(\frac{R}{2\pi}\right)^{1+b^2} \psi\bar{\psi} \cosh(b\phi) + \pi\mu^2 b^2 \left(\frac{R}{2\pi}\right)^{2+2b^2} [\cosh(2b\phi) - 1] \Bigg] \end{aligned} \tag{81}$$

In the UV limit, the exponential potential becomes negligible except in the region where ϕ_0 goes to $\pm\infty$. This means that the SShG model is effectively described by the super-LFT as $R \to 0$. From the ground state energy for the primary state labelled by P, the effective central charge can be obtained by

$$\begin{aligned} c_{\text{eff}}(R) &= \tfrac{3}{2} - 12P^2 + \mathcal{O}(R) \qquad (\text{NS}) \\ &= -12P^2 + \mathcal{O}(R) \qquad (\text{R}) \end{aligned} \tag{82}$$

For the (NS) sector, P corresponding to the ground state is determined again by the quantization condition coming from the super-LFT reflection amplitudes:

$$\delta_{\text{NS}}(P) = \pi + 2QP \ln \frac{R}{2\pi} \tag{83}$$

where $\delta_{\rm NS}(P)$ is the phase factor of (NS) reflection amplitudes. This quantization condition can be solved iteratively by expanding $\delta_{\rm NS}(P)$ in powers of P,

$$\begin{aligned}\delta_{\rm NS}(P) &= \delta_1^{\rm NS} P + \delta_3^{\rm NS} P^3 + \delta_5^{\rm NS} P^5 + \cdots \\ \delta_1^{\rm NS} &= -2\left\{\frac{1}{b}\ln\left[\frac{\pi\mu}{2}\gamma\left(\frac{1+b^2}{2}\right)\right] + \gamma_{\rm E} Q\right\} \\ \delta_3^{\rm NS} &= \frac{2}{3}\zeta(3)\left(b^3+\frac{1}{b^3}\right) \\ \delta_5^{\rm NS} &= -\frac{2}{5}\zeta(5)\left(b^5+\frac{1}{b^5}\right)\end{aligned} \tag{84}$$

The (R) sector shows very different behaviour from the (NS). The physical meaning becomes clear if one considers the $P\to 0$ limit where $S_{\rm R}(P)\to 1$ comparing with $S_{\rm NS}(P)\to -1$. While for the (NS) sector $\Psi_P \sim 2iP\phi_0$ so that the quantum number n should be 1 as in Eq. (83), the wave functional for the (R) sector becomes constant corresponding to $n=0$. Therefore, the quantization condition becomes

$$\delta_{\rm R}(P) = 2QP\ln\frac{R}{2\pi} \tag{85}$$

An obvious solution is $P=0$ so that

$$c_{\rm eff}(R) = 0 + \mathcal{O}(R) \tag{86}$$

In the $b\to 0$ limit, one can verify this from the (R) sector zero-mode dynamics of the SShG model which is governed by the Hamiltonain

$$H_0^{\rm R} = -\left(\frac{\partial}{\partial\phi_0}\right)^2 + 4\pi^2\mu^2 b^2\sinh^2 b\phi_0 + 4\pi i\mu b^2\psi_0\bar{\psi}_0\cosh b\phi_0 \tag{87}$$

This is a typical supersymmetric quantum mechanics problem and in general there exists a zero-energy ground state[(23)] if the supersymmetry is not broken. Explicitly, the wavefunction of the state is found to be

$$\Psi_0(\phi_0) = \begin{pmatrix} e^{-2\pi\mu\cosh b\phi_0} \\ 0 \end{pmatrix} \tag{88}$$

This state is normalizable and its energy is exactly zero. Thus at least in $b\to 0$ limit, $c_{\rm eff}$ is exactly zero regardless of r without any power correction.

5. COMPARISON WITH THE TBA RESULTS

A standard approach to study the scaling behaviour of integrable QFTs is to solve the TBA equations. In this section we compute the scaling functions in the UV region from the TBA equations and compare them with the results in the previous section based on the reflection amplitudes.

5.1. TBAs of ATFTs and BD Model

The TBA equations for the ATFTs are given by $(i=1,\dots,r)$

$$c_{\text{eff}}^{(\text{TBA})}(R)=\sum_{i=1}^{r}\frac{3Rm_i}{\pi^2}\int\cosh\theta\log(1+e^{-\varepsilon_i(\theta,R)})\,d\theta \tag{89}$$

where m_i's are particles masses and functions $\varepsilon_i(\theta, R)$ $(i=1,\dots,r)$ satisfy the system of r coupled integral equations:

$$m_iR\cosh\theta=\varepsilon_i(\theta,R)+\sum_{j=1}^{r}\int\varphi_{ij}(\theta-\theta')\log(1+e^{-\varepsilon_i(\theta',R)})\frac{d\theta'}{2\pi} \tag{90}$$

with the kernels φ_{ij}, equal to the logarithmic derivatives of the S-matrices $S_{ij}(\theta)$ of ATFTs,[24–27]

$$\varphi_{ij}(\theta)=-i\frac{d}{d\theta}\log S_{ij}(\theta)$$

Equation (90) becomes the TBA equation of BD model when $r=1$ and the kernel is given by

$$\varphi(\theta)=\Phi_{2/3}(\theta)+\Phi_{-B/3}(\theta)+\Phi_{(B-2)/3}(\theta),\qquad\text{with}\qquad B=\frac{b^2}{1+b^2/2}$$

where

$$\Phi_x(\theta)\equiv\frac{4\sin\pi x\cosh\theta}{\cos 2\pi x-\cosh 2\theta} \tag{91}$$

The function $E^{(\text{TBA})}(R)$ defined from the TBA equations differs from the ground state energy $E(R)$ of the system on the circle of size R by the bulk term: $E^{(\text{TBA})}(R)=E(R)-fR$, where f is a specific bulk free energy.[3] To compare the same functions we should subtract this term from the function $E(R)$ defined by Eq. (70) i.e.,

$$c_{\text{eff}}^{(\text{TBA})}(R)=C_{\text{eff}}^{(\text{RA})}(R)+\frac{6R^2}{\pi}f(G) \tag{92}$$

The specific bulk free energy $f(G)$ can be calculated by Bethe Ansatz method. For ATFTs,[28, 29, 9, 10]

$$
\begin{aligned}
f(G) &= \frac{\bar{m}^2 \sin(\pi/h)}{8 \sin(\pi B/h) \sin(\pi(1-B)/h)}, & G &= \text{ADE series} \\
f(G) &= \frac{\bar{m}^2 \sin(\pi/H)}{8 \sin(\pi B/H) \sin(\pi(1-B)/H)}, & G &= B_r^{(1)},\, C_r^{(1)} \\
f(G) &= \frac{\bar{m}^2 \cos(\pi(1/3 - 1/H))}{16 \cos(\pi/6) \sin(\pi B/H) \sin(\pi(1-B)/H)}, & G &= G_2^{(1)},\, F_4^{(1)}
\end{aligned}
\tag{93}
$$

where

$$
B = \frac{b^2}{1+b^2}, \qquad H = \frac{h + b^2 h^{\vee}}{1+b^2} \tag{94}
$$

For the BD model,[19]

$$
f = \frac{m^2}{16\sqrt{3}\sin(\pi B/6)\sin(\pi(2-B)/6)} \tag{95}
$$

The contribution of bulk term $f(G)$ becomes quite essential at $R \sim \mathcal{O}(1)$.

The TBA equations (90) are solved numerically for various algebras. The effective central charge $c_{\text{eff}}^{(\text{TBA})}(R)$ is then computed from Eq. (89) for many different values of parameter $\bar{m}R$. After taking into account the bulk term, the numerical solution for $c_{\text{eff}}^{(\text{TBA})}(R)$ is fitted with the expansions (70) in $1/l$ considering the coeffiecients as fitting parameters.

To compare the numerical TBA results with analytical ones from reflection amplitudes, we need to know the exact relations between parameters μ_i of the action and masses of particles m_i. This is because TBA equations are derived from S-matrix data while the method of reflection amplitudes deals with the paramters of the action directly. These "mass-μ" relations[18, 29, 9] are given in the Appendix. With the help of them, we can express $c_{\text{eff}}(R)$ obtained in the previous section purely in terms of particles masses m_i. For example, the function $L(R)$ of ATFTs defined in Eq. (64) becomes

$$
L = -\frac{2}{b}(h + b^2 h^{\vee}) \ln\left[\frac{\bar{m}R}{4\pi} k(G)\, \Gamma\left(\frac{1-B}{H}\right) \Gamma\left(1 + \frac{B}{H}\right)\right] + \frac{2}{b}\ln(b^{2h}(\xi^2/2)^z) \tag{96}
$$

Similarly we can rewrite μ and μ' in (78) in terms of the particle mass m as,

$$\bar{\delta}_1 = \frac{6}{b}\ln\frac{b^2}{2} - 2(Q+Q')\left[\ln\frac{m\Gamma(1+b^2/(6+3b^2))\,\Gamma(2/(6+3b^2))}{2\sqrt{3}\,\Gamma(1/3)} + \gamma_{\mathrm{E}}\right] \tag{97}$$

Comparing the results, we found that, up to the order $1/l^7$, the numerical TBA results are in excellent agreement with the analytic results given in previous sections. (For the details of the comparison, see refs. 8 and 9.) To see the agreement more concretely, we plot the functions $c_{\mathrm{eff}}^{(\mathrm{TBA})}(R)$ and $c_{\mathrm{eff}}^{(\mathrm{RA})}(R)$ for non-simply laced ATFTs setting $B=0.5$. The first function is computed numerically from TBA equations. The second one is calculated using Eqs. (63) and (66), based on the reflection amplitudes, with taking into account the bulk free energy term according to Eq. (92). Figure 2 shows that, for all models, the two curves are almost identical without essential difference in the graphs even at $R\sim\mathcal{O}(1)$. This good agreement outside the UV region looks not to be accidental. However, at present, we have no satisfactory explanation of this interesting phenomena in ATFTs.

5.2. TBA of SShG Model

Finally we briefly describe the TBA analysis of SShG model (see ref. 7 for details). There are two sectors in SShG model and the corresponding TBA equations are different from each other. They can be written as[7]

$$c_{\mathrm{eff}}(r) = \frac{3r}{\pi^2}\int \cosh\theta\,\ln(1\pm e^{-\varepsilon_1(\theta)})\,d\theta \tag{98}$$

where the pseudo-energies are the solution of the equations,

$$\begin{aligned} \varepsilon_1(\theta) &= r\cosh\theta - \int\frac{d\theta'}{2\pi}\,\varphi(\theta-\theta')\ln[1\pm e^{-\varepsilon_2(\theta')}] \\ \varepsilon_2(\theta) &= -\int\frac{d\theta'}{2\pi}\,\varphi(\theta-\theta')\ln[1\pm e^{-\varepsilon_1(\theta')}] \end{aligned} \tag{99}$$

In the above equations, the plus (minus) sign corresponds to the NS (R) sector and the kernel is given by

$$\varphi(\theta) = -\Phi_B(\theta) \qquad \text{with} \qquad B = \frac{b^2}{1+b^2}$$

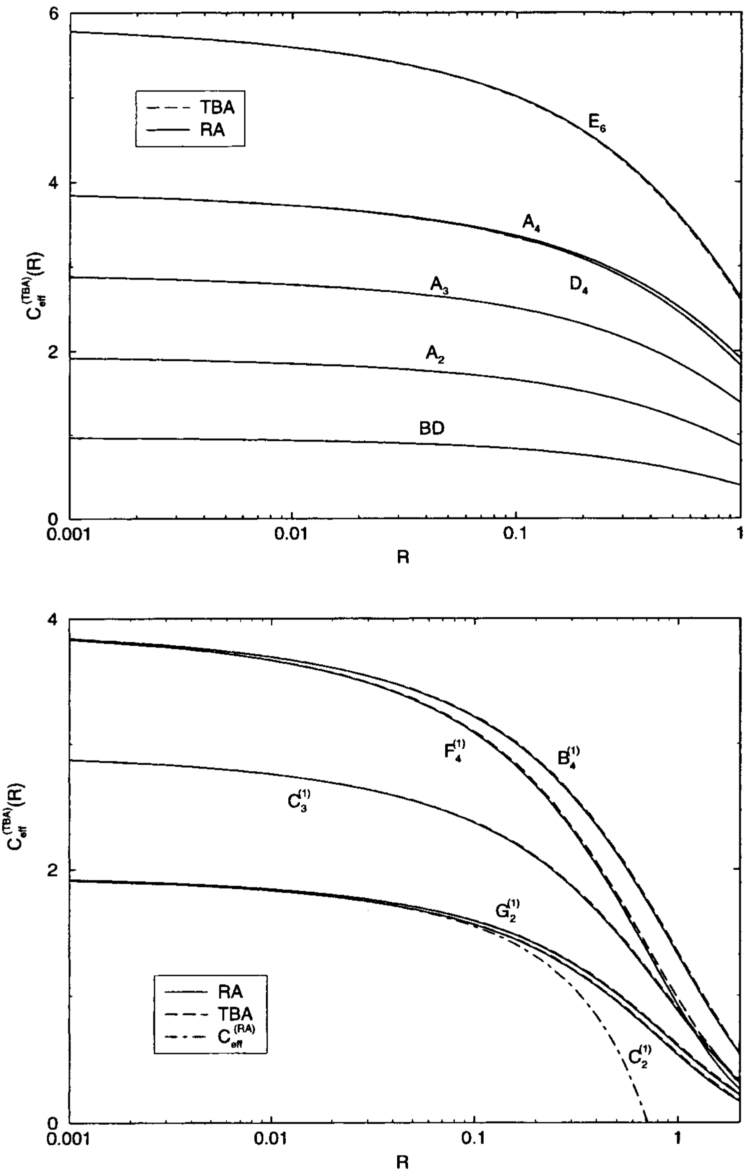

Fig. 2. (a) Plot of c_{eff} for A_2, A_3, A_4, D_4, A_6, E_6 ATFTs and BD model at $B=0.5$. (b) Plot of $c_{\text{eff}}^{(\text{TBA})}$ for $C_2^{(1)}$, $C_3^{(1)}$, $B_4^{(1)}$, $G_2^{(1)}$ and $F_4^{(1)}$ ATFTs at $B=0.5$. As an example, we also display $C_{\text{eff}}^{(\text{RA})}$ for $C_2^{(1)}$ calculated without taking into account the bulk term. The difference between this function and $c_{\text{eff}}^{(\text{TBA})}$ gives the bulk free energy of $C_2^{(1)}$ ATFT according to Eq. (63).

where $\Phi_B(\theta)$ is defined in Eq. (91). In the case of NS sector, we can perform the numerical analysis following the method described before and again we have an excellent agreement between the results of TBA and reflection amplitudes. On the other hand, we find that the R sector TBA equation is trivial, giving vanishing $c_{\text{eff}}(R)$. This is consistent with the existence of the supersymmetric zero mode as explained in Section 4.3.

6. VACUUM EXPECTATION VALUE OF EXPONENTIAL OPERATOR

The reflection amplitude, being the quantity derived from CFT, plays a crucial role in the calculation of one-point functions in perturbed CFT. In this section, we will demonstrate how powerful this method is by constructing explicitly the one-point funtion of ATFTs. One point function $G(\boldsymbol{a})$ we are considering is defined as the vacuum expectaion value of the vertex operator $V_{\boldsymbol{a}}(x) = \exp(\boldsymbol{a} \cdot \boldsymbol{\varphi}(x))$ of the ATFTs (56):

$$G(\boldsymbol{a}) = \langle V_{\boldsymbol{a}}(x) \rangle = \langle \exp(\boldsymbol{a} \cdot \boldsymbol{\varphi}(x)) \rangle \tag{100}$$

We start with the one point function of the simplest case, sinh-Gordon model, $A_1^{(1)}$ ATFT[11, 12] and Bullough–Dodd model, $A_2^{(2)}$. This approach is generalized to ATFTs.[13, 14, 8, 9] This same approach was also applied to two-parameter family of integrable models.[15]

6.1. One-Point Function of Sinh-Gordon Model and Bullough–Dodd Model

The sinh-Gordon model (45) has the one component field, which can be considered as the perturbed LFT. The operator $V_{sG}(x) = \exp(\sqrt{2}\, a\phi(x))$ satisfies the "reflection relation"

$$\exp(\sqrt{2}\, a\phi(x)) = R_{sG}(a) \exp(\sqrt{2}(Q-a)\, \phi(x)) \tag{101}$$

We put $\boldsymbol{a} \to \sqrt{2}\, a$ in the vertex operator for convenience. $R_{sG}(a)$ is given in terms of Liouville reflection amplitude, $S_L(\sqrt{2}\, P)$ in Eq. (23).

The vacuum expectation value $G_{sG}(a)$ of the sinh-Gordon model is considered to satisfy the same reflection relation,

$$G_{sG}(a) = R_{sG}(a)\, G_{sG}(Q-a) \tag{102}$$

In addition, since the sinh-Gordon model is invariant under the parity transformation $\phi \to -\phi$, the one-point function is an even function of a.

$$G_{sG}(-a) = G_{sG}(a) \tag{103}$$

This symmetric consideration determines $G_{sG}(a)$ up to a periodic function.

The one-point function of the sinh-Gordon model is given as the minimal solution of Eqs. (102), (103).

$$G_{sG}(a) = (-\pi\mu\gamma(1+b^2))^{-a^2/(1+b^2)} \times \exp \int \frac{dt}{t} (2a^2 e^{-2t} - \mathscr{F}(a,t)) \tag{104}$$

with

$$\mathscr{F}(a,t) = \left[\frac{\sinh^2(2abt)}{2\sinh(t)\sinh(b^2 t)\sinh((1+b^2)t)} \right] \tag{105}$$

The result is checked by various consideration including the classical equation, free energy of the theory and perturbative calculation.[11]

For the Bullough–Dodd model (74), the one-point function $G_{\mathrm{BD}}(x) = \langle \exp(2a\phi(x)) \rangle$ is similarly found.[13] (Here the factor 2 is included in the exponential again for convenience. This has to be related with the normalization of the action.)

$$G_{\mathrm{BD}}(a) = R_{\mathrm{BD}}(a)\, G_{\mathrm{BD}}(Q-a) \tag{106}$$

where $R_{\mathrm{BD}} = S_L(2P)$. However, the parity symmetry is no longer the symmetry of BD. Instead, we have

$$G_{\mathrm{BD}}(a) = R'_{\mathrm{BD}}(a)\, G_{\mathrm{BD}}(Q'-a) \tag{107}$$

where Q' is given in Eq. (76) and $R'_{\mathrm{BD}}(a) = S'_L(2P)$ as in Eq. (75). From this one obtains the one-point function of the BD model as

$$\begin{aligned} G_{\mathrm{BD}}(a) = {} & \left[\frac{\mu'}{\mu} \frac{2^{b^2/2}\Gamma(1-b^2)\,\Gamma(1+b^2/4)}{\Gamma(1+b^2)\,\Gamma(1-b^2/4)} \right]^{2a/3b} \\ & \times \left[\frac{m\Gamma(1+b^2/(6+3b^2))\,\Gamma(2/(6+3b^2))}{2^{2/3}\sqrt{3}\,\Gamma(1/3)} \right]^{ab-2a^2} \\ & \times \exp\left[\int_0^{+\infty} \frac{dt}{t} \left(-\frac{\sinh((2+b^2)t)\,\Psi(t,a)}{\sinh(3(2+b^2)t)\sinh(2t)\sin h(b^2 t)} + 2a^2 e^{-2t} \right) \right] \end{aligned} \tag{108}$$

where

$$\Psi(t, a) = \sinh(2abt)[\sinh((4+b^2+2ab)\,t) - \sinh((2+2b^2-2ab)\,t) + \sinh((2+b^2+2ab)\,t) - \sinh((2+b^2-2ab)\,t) - \sinh((2-b^2+2ab)\,t)]$$

and m is given in Appendix.

6.2. One-Point Function in Simply-Laced ATFTs

Simply laced ATFTs can be considered as the obvious generalization of the sinh-Gordon model. The complexity comes from the fact that more than two operators are of the same conformal dimension and satisfy the reflection relation.[14, 8] The Weyl reflection of Lie algebra space takes over the simple parity reflection in sinh-Gordon model. We summarize here the general rule for the one-point function to satisfy the requirements:

- [R1] Analytic property: $G(\boldsymbol{a})$ is meromorphic in $\boldsymbol{a}$.
- [R2] Normalization: $G(\boldsymbol{a}=0)=1$.
- [R3] Reflection relation from (8): $G(\mathbf{Q}+i\mathbf{P}) = R_s(\boldsymbol{a})\,G(\mathbf{Q}+is\mathbf{P})$
- [R4] Symmetry of the system: $G(\tau\boldsymbol{a}) = G(\boldsymbol{a})$. τ is the symmetry operation of the action.

The minimal solution satisfying these requirements is given in refs. 14 and 8

$$G(\boldsymbol{a}) = (-\pi\mu\gamma(1+b^2))^{-a^2/(2(1+b^2))} \times \exp\int\frac{dt}{t}\,(\boldsymbol{a}^2 e^{-2t} - \mathscr{F}(\boldsymbol{a}, t)) \tag{109}$$

with

$$\mathscr{F}(\boldsymbol{a}, t) = \left[\frac{\sinh((b^2+1)\,t)}{\sinh(t)\,\sinh(b^2 t)\,\sinh((1+b^2)\,ht)}\right] \times I(\boldsymbol{a}, t) \tag{110}$$

and

$$I(\boldsymbol{a}, t) = \sum_{\boldsymbol{\alpha}>0}\,[\sinh(\boldsymbol{\alpha}\cdot\boldsymbol{a}bt)\,\sinh((\boldsymbol{\alpha}\cdot\boldsymbol{a}b - 2\boldsymbol{\alpha}\cdot\mathbf{Q}b + h(1+b^2)\,t)] \tag{111}$$

One may simply prove that [R1] and [R2] are satisfied. Requirement [R3] is checked by confirming the result for the Weyl reflection s_i with respect to any simple root $\mathbf{e}_i$:

$$G(\mathbf{Q}+i\mathbf{P}_i) = S_L(\mathbf{e}_{i_1}, \mathbf{P})\,G(\mathbf{Q}+is_i\mathbf{P}) \tag{112}$$

where $P_i = \mathbf{P} \cdot \mathbf{e}_i$. Here are two ingredients to be checked, $\boldsymbol{a}^2$ and $I(\boldsymbol{a}, t)$, which under the reflection, result in

$$(\mathbf{Q} + i\mathbf{P})^2 - (\mathbf{Q} + is_i\mathbf{P})^2 = 2i\left(b + \frac{1}{b}\right)\mathbf{P}_i = 2iQ\mathbf{P}_i \tag{113}$$

and

$$\begin{aligned} I(\boldsymbol{a}, t) &= I(\mathbf{Q} + i\mathbf{P}) - I(\mathbf{Q} + is_i\mathbf{P}) \\ &= \tfrac{1}{2}(\cosh[2biP_i t + h(1+b^2)\,t] - \cosh[-2biP_i t + h(1+b^2)\,t]) \\ &= \sinh(2biP_i t)\sinh h(1+b^2)\,t \end{aligned} \tag{114}$$

Combining this two reflection property, $G(\boldsymbol{a})$ satisfies the Eq. (112).

The symmetry operation τ in the the requirement [R4] is the Dynkin diagram symmetry of ADE series. Obviously $\boldsymbol{a}^2$ is invariant under τ. One needs to prove that $I(\boldsymbol{a}, t)$ is invariant. Let us rearrange $I(\boldsymbol{a}, t)$ in Eq. (110) as

$$I(\boldsymbol{a}, t) = \tfrac{1}{2}[J(\boldsymbol{a}, t) - J(\boldsymbol{a}_i = 0, t)] \tag{115}$$

where

$$J(\boldsymbol{a}, t) = \sum_{\boldsymbol{a} > 0} \cosh(2b(\boldsymbol{a} \cdot \boldsymbol{\alpha} - \mathbf{Q} \cdot \boldsymbol{\alpha})\,t + h(1+b^2)\,t) \tag{116}$$

The problem reduces to prove $J(\boldsymbol{a}, t)$ invariant under τ. Under τ, if a positive root goes to another positive root, this only reshuffles the terms in $J(\boldsymbol{a}, t)$.

However, there are cases where a postive root goes to a negative root, $-\boldsymbol{\beta} = \tau\boldsymbol{\alpha}$. Then apart from the reshuffling, $J(\boldsymbol{a}, t)$ contains the term,

$$\cosh(2b\boldsymbol{a} \cdot (-\boldsymbol{\beta})\,t - 2b\mathbf{Q} \cdot \boldsymbol{\alpha}t + h(1+b^2)\,t) \tag{117}$$

This is the case when τ changes a simple root $\boldsymbol{\alpha}_\tau$ to zeroth root $\mathbf{e}_0$: $\tau\boldsymbol{\alpha}_\tau = \mathbf{e}_0$. Noting that in ADE series, any postive root can contain at most one such a root $\boldsymbol{\alpha}_\tau$, and the root satisfies an identity,

$$\boldsymbol{\rho} \cdot (\boldsymbol{\alpha} - \tau\boldsymbol{\alpha}) = \boldsymbol{\rho} \cdot (\boldsymbol{\alpha}_\tau - \mathbf{e}_0) = h \tag{118}$$

From this one can find a unique positive root $\boldsymbol{\alpha}$ satisfying

$$2b\mathbf{Q} \cdot \boldsymbol{\alpha} - h(1+b^2) = -2b\mathbf{Q} \cdot \boldsymbol{\beta} + h(1+b^2) \tag{119}$$

and therefore, $J(\boldsymbol{a}, t)$ is invariant under τ.

6.3. One-Point Function in Nonsimply-Laced ATFTs

One-point function of non-simply laced ATFTs can be obtained in the same way as follows:[9, 10]

$$G(\boldsymbol{a}) = K(\boldsymbol{a}) \times \exp \int \frac{dt}{t} \left(\boldsymbol{a}^2 e^{-2t} - \mathscr{F}(\boldsymbol{a}, t)\right) \tag{120}$$

where

$$K(\boldsymbol{a}) = \prod_{\boldsymbol{\alpha}>0} \left[\left(-\pi\mu_{\boldsymbol{\alpha}}\gamma \left(1 + \frac{b^2\, |\boldsymbol{\alpha}|^2}{2} \right) \right)^{-((\boldsymbol{\alpha}\cdot\boldsymbol{a})(\boldsymbol{\alpha}\cdot\boldsymbol{a}b - 2\boldsymbol{\alpha}\cdot\mathbf{Q}b + HQb))/(|\boldsymbol{\alpha}|^2\, HQb^2)} \right] \tag{121}$$

and

$$\mathscr{F}(\boldsymbol{a}, t) = \sum_{\boldsymbol{\alpha}>0} \left[\frac{\begin{pmatrix} \sinh(\boldsymbol{\alpha}\cdot\boldsymbol{a}bt)\sinh((\boldsymbol{\alpha}\cdot\boldsymbol{a}b - 2\boldsymbol{\alpha}\cdot\mathbf{Q}b + HQb)\, t) \\ \times \sinh(((b^2\, |\boldsymbol{\alpha}|^2/2) + 1)\, t) \end{pmatrix}}{\sinh(t)\sinh((b^2\, |\boldsymbol{\alpha}|^2/2)\, t)\sinh(HQbt)} \right] \tag{122}$$

where $Qb = 1 + b^2$, $B = b^2/(1 + b^2)$ and $H = h(1 - B) + h^\vee B$.

It is simple to show that $G(\boldsymbol{a})$ in Eq. (120) satisfies the requirements [R1] and [R2]. For the requirements of [R3], we can proceed same as in the simply laced case for $\boldsymbol{\alpha}^2$ and $\mathscr{F}$ parts. Only the factor $K(\boldsymbol{\alpha})$ needs care. The exponent of the factor K

$$L_{\boldsymbol{\alpha}}(\boldsymbol{a}) = (\boldsymbol{\alpha}\cdot\boldsymbol{a})(\boldsymbol{\alpha}\cdot\boldsymbol{a} - 2\boldsymbol{\alpha}\cdot\mathbf{Q} + HQ) \tag{123}$$

is rewritten as

$$L_{\boldsymbol{\alpha}}(\boldsymbol{a}) = \left(\boldsymbol{a}\cdot\boldsymbol{\alpha} - \left(\boldsymbol{\alpha}\cdot\mathbf{Q} - \frac{HQ}{2} \right) \right)^2 - \left(\boldsymbol{\alpha}\cdot\mathbf{Q} - \frac{HQ}{2} \right)^2 \tag{124}$$

Under the Weyl reflection, one has

$$L_{\boldsymbol{\alpha}}(\mathbf{Q} + i\mathbf{P}) - L_{\boldsymbol{\alpha}}(\mathbf{Q} + i\hat{s}_i\mathbf{P}, \boldsymbol{\alpha}) = i\mathbf{P}\cdot(\boldsymbol{\alpha} - \hat{s}_i\boldsymbol{\alpha}) \tag{125}$$

Now the Weyl reflection changes positive roots to another positive root with the same length except the simple root $\boldsymbol{\alpha}_i$, $\hat{s}_i\boldsymbol{\alpha}_i = -\boldsymbol{\alpha}_i$, we have the ration of $K(\boldsymbol{a})$'s as

$$\frac{K(\mathbf{Q} + i\mathbf{P})}{K(\mathbf{Q} + i\hat{s}_i\mathbf{P})} = \left(-\pi\mu_i\gamma \left(1 + \frac{b^2\boldsymbol{\alpha}_i^2}{2} \right) \right)^{-(\mathbf{P}\cdot\boldsymbol{\alpha}_i)/b\boldsymbol{\alpha}_i^2} \tag{126}$$

which proves the requirement [R3].

Finally, if the Dynkin diagram symmetry operation τ changes a positive root to another positive root, this will reshuffle the multiplication factors in $G(\boldsymbol{\alpha})$. On the other hand, When $\tau\boldsymbol{\alpha}$ becomes a negative root $-\beta = \tau\boldsymbol{\alpha}$, one may apply the similar argument of the simply laced case and find $G(\boldsymbol{\alpha})$ is invariant. Here one needs the identities, $(\boldsymbol{\alpha}+\boldsymbol{\beta})\cdot\boldsymbol{\rho}^\vee = h$ and $(\boldsymbol{\alpha}+\boldsymbol{\beta})\cdot\boldsymbol{\rho} = h^\vee$.

Of course, the symmetry operation requirements [R1]–[R4] do not guarantee that $G(\boldsymbol{\alpha})$ is the correct one-point function. The solution we have is the minimal solution which is meromorphic function of $\boldsymbol{\alpha}$. In fact, one can show that this one-point function $G(\boldsymbol{\alpha})$ does give the correct bulk free energy in Eq. (93).[9] The one-point function of the BD model introduced earlier can be reproduced since it is the dual theory of nonsimply-laced ATFTs, $A_2^{(2)}$. Furthermore, one may check that the result is consistent with the perturbative calculation.[10]

APPENDIX. MASS-μ RELATIONS

In the BD model, the parameters μ and μ' in the action are related to the mass of on-shell particle by[19]

$$m = \frac{2\sqrt{3}\,\Gamma(1/3)}{\Gamma(1+b^2/(6+3b^2))\,\Gamma(2/(6+3b^2))}\left[-\frac{\mu\pi\Gamma(1+b^2)}{\Gamma(-b^2)}\right]^{1/(6+3b^2)} \times\left[-\frac{2\mu'\pi\Gamma(1+b^2/4)}{\Gamma(-b^2/4)}\right]^{2/(6+3b^2)} \tag{127}$$

For SShG model, $m-\mu$ relation is given in ref. 15,

$$\frac{\pi}{2}\mu b^2\gamma\left(\frac{1+b^2}{2}\right) = \left[\frac{m}{8}\,\frac{\pi b^2/(1+b^2)}{\sin(\pi b^2/(1+b^2))}\right]^{1+b^2} \tag{128}$$

The spectrum of simply laced ATFTs consists of r particles with the masses m_i $(i=1,\dots,r)$ given by

$$m_i = \bar{m}\nu_i \tag{129}$$

where

$$\bar{m}^2 = \frac{1}{2h}\sum_{i=1}^{r} m_i^2 \tag{130}$$

and v_i^2 are the eigenvalues of the mass matrix:

$$M_{ab} = \sum_{i=1}^{r} n_i (\mathbf{e}_i)^a (\mathbf{e}_i)^b + (\mathbf{e}_0)^a (\mathbf{e}_0)^b \tag{131}$$

The parameter $\bar{m}$ characterizing the spectrum of physical particles can be related with the parameter μ in the action using Bethe ansatz method[18, 29] and the result is

$$-\pi\mu\gamma(1+b^2) = \left[\frac{\bar{m}k(G)\, \Gamma(1/(1+b^2)\, h)\, \Gamma(1+(b^2/(1+b^2)\, h))}{2\Gamma(1/h)}\right]^{2(1+b^2)} \tag{132}$$

where

$$k(G) = \left(\prod_{i=1}^{r} n_i^{n_i}\right)^{1/2h} \tag{133}$$

In the case of non-simply laced ATFTs, the exact mass ratios are different from the classical ones and get quantum corrections.[26, 27] Using the notation (94), the spectrum of ATFTs are expressed in terms of one mass parameter $\bar{m}$ as:

$$\begin{aligned}
&B_r^{(1)}: M_r = \bar{m}, \qquad M_a = 2\bar{m}\sin(\pi a/H), \qquad a = 1, 2,..., r-1\\
&C_r^{(1)}: M_a = 2\bar{m}\sin(\pi a/H), \qquad a = 1, 2,..., r\\
&G_2^{(1)}: M_1 = \bar{m}, \qquad M_2 = 2\bar{m}\cos(\pi(1/3 - 1/H))\\
&F_4^{(1)}: M_1 = \bar{m}, \qquad M_2 = 2\bar{m}\cos(\pi(1/3 - 1/H))\\
&\qquad M_3 = 2\bar{m}\cos(\pi(1/6 - 1/H)), \qquad M_4 = 2M_2\cos(\pi/H)
\end{aligned} \tag{134}$$

Again, $\bar{m}$ can be written as a function of the parameter μ_i as,[9]

$$\prod_{i=0}^{r} \left[-\pi\mu_i\gamma(1+\mathbf{e}_i^2 b^2/2)\right]^{n_i} = \left[\frac{\bar{m}k(G)}{2}\, \Gamma\left(\frac{1-B}{H}\right) \Gamma\left(1+\frac{B}{H}\right)\right]^{2H(1+b^2)} \tag{135}$$

where $k(G)$ is a function depending on the algebra:

$$\begin{aligned}
k(B_r^{(1)}) &= \frac{2^{-2/H}}{\Gamma(1/H)}, \qquad k(C_r^{(1)}) = \frac{2^{2B/H}}{\Gamma(1/H)}\\
k(G_2^{(1)}) &= \frac{\Gamma(2/3)}{2\Gamma(1/2)\,\Gamma(1/6+1/H)}, \qquad k(F_4^{(1)}) = \frac{\Gamma(2/3)}{2\Gamma(1/2)\,\Gamma(1/6+1/H)}
\end{aligned} \tag{136}$$

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REFERENCES

1. A. B. Zamolodchikov, *Int. J. Mod. Phys. A* **4**:4235 (1989).
2. C. N. Yang and C. P. Yang, *J. Math. Phys.* **10**:1115 (1969).
3. Al. B. Zamolodchikov, *Nucl. Phys. B* **342**:695 (1990).
4. Al. B. Zamolodchikov, *Resonance Factorized Scattering and Roaming Trajectories*, preprint ENS-LPS-335 (1991).
5. V. A. Fateev, E. Onofri, and Al. B. Zamolodchikov, *Nucl. Phys. B* **406**:521 (1993).
6. A. B. Zamolodchikov and Al. B. Zamolodchikov, *Nucl. Phys. B* **477**:577 (1996).
7. C. Ahn, C. Kim, and C. Rim, *Nucl. Phys. B* **556**:505 (1999).
8. C. Ahn, V. Fateev, C. Kim, C. Rim, and B. Yang, *Nucl. Phys. B* **565**:611 (2000).
9. C. Ahn, P. Baseilhac, V. Fateev, C. Kim, and C. Rim, *Phys. Lett. B* **481**:114 (2000).
10. C. Ahn, P. Baseilhac, C. Kim, and C. Rim, in preparation.
11. S. Lukyanov and A. Zamolodchikov, *Nucl. Phys. B* **493**:571 (1997).
12. V. Fateev, S. Lukyanov, A. Zamolodchikov, and Al. Zamolodchikov, *Phys. Lett. B* **406**:83 (1997).
13. V. Fateev, S. Lukyanov, A. Zamolodchikov, and Al. Zamolodchikov, *Nucl. Phys. B* **516**:562 (1998).
14. V. Fateev, *Mod. Phys. Lett. A* **15**:259 (2000).
15. P. Baseilhac and V. A. Fateev, *Nucl. Phys. B* **532**:567 (1998); Al. B. Zamolodchikov, in private communications.
16. V. Fateev and S. Lukyanov, *Sov. Sci. Rev. A* **212**:212 (Physics) (1990).
17. P. Bouwknegt and K. Schoutens, *W-symmetry* (World Scientific, Singapore, 1995).
18. Al. B. Zamolodchikov, *Int. J. Mod. Phys. A* **10**:1125 (1995).
19. V. Fateev, S. Lukyanov, A. Zamolodchikov, and Al. Zamolodchikov, *Phys. Lett. B* **406**:83 (1997).
20. R. C. Rashkov and M. Stanishkov, *Phys. Lett. B* **380**:49 (1996).
21. R. H. Poghossian, *Nucl. Phys. B* **496**:451 (1997).
22. T. Curtright and G. Ghandour, *Phys. Lett. B* **136**:50 (1984).
23. E. Witten, *Nucl. Phys. B* **185**:513 (1981).
24. A. Arinstein, V. Fateev, and A. Zamolodchikov, *Phys. Lett. B* **87**:389 (1979).
25. H. W. Braden, E. Corrigan, P. E. Dorey, and R. Sasaki, *Nucl. Phys. B* **338**:689 (1990).
26. G. W. Delius, M. T. Grisaru, S. Penati, and D. Zanon, *Phys. Lett. B* **256**:164 (1991); C. Destri, H. de Vega, and V. Fateev, *Phys. Lett. B* **256**:173 (1991).
27. E. Corrigan, P. E. Dorey, and R. Sasaki, *Nucl. Phys. B* **408**:579 (1993).
28. C. Destri and H. de Vega, *Nucl. Phys. B* **358**:251 (1991).
29. V. A. Fateev, *Phys. Lett. B* **324**:45 (1994).