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EXACT S-MATRICES FOR AdS_3/CFT_2

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We propose exact S-matrices for the AdS_3/CFT_2 duality between type IIB strings on $AdS_3 \times S^3 \times M_4$ with $M_4 = S^3 \times S^1$ or T^4 and the corresponding two-dimensional conformal field theories. We fix the two-particle S-matrices on the basis of the symmetries $su(1|1)$ and $su(1|1) \times su(1|1)$. A crucial justification comes from the derivation of the all-loop Bethe ansatz matching exactly the recent conjecture proposed by Babichenko *et al.* [*J. High Energy Phys.* **1003**, 058 (2010), arXiv:0912.1723 [hep-th]] and Ohlsson Sax and Stefanski, Jr. [*J. High Energy Phys.* **1108**, 029 (2011), arXiv:1106.2558 [hep-th]].

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1. Introduction

The discovery of integrable structures on both sides of the AdS_5/CFT_4 correspondence,[3,4] was crucial in understanding and determining exactly some important physical quantities (see Ref. 5 and references therein) in $\mathcal{N} = 4$ super-Yang–Mills and the IIB superstring theory on $AdS_5 \times S^5$ in the planar limit.

From the integrability point of view, one of the most recently investigated examples of such gauge/string duality is the AdS_3/CFT_2 correspondence between IIB superstring theory on $AdS_3 \times S^3 \times S^3 \times S^1$ or $AdS_3 \times S^3 \times T^4$ backgrounds with RR fluxes and yet quite unknown two-dimensional CFTs.[a] Indeed, while the

[a]In the case of $AdS_3 \times S^3 \times T^4$, the CFT dual is a $\mathcal{N} = (4, 4)$ theory on a symmetric product of T^4.[6–8]

NS AdS_3/CFT_2 was solved completely by implementing techniques typical of two-dimensional CFTs,[9–16] the RR counterpart remains quite obscure: there the usual 2D CFT methods fail, then one of the most promising way to tackle this problem is given by integrability techniques.

This investigation started in Refs. 1 and 2, where a set of all-loop Bethe equations, describing in principle at any coupling the asymptotic spectrum of the string energies and the dimensions of the yet unknown gauge operators, were proposed on the basis of classical integrability of the corresponding supercoset sigma models.[b] Unfortunately, this approach cannot take into account the contribution of some (massless) modes of the full string theory. Some progress in the direction of incorporating them has been done very recently in Ref. 18, where a set of Bethe equations has been proposed in order to describe the massless modes sector. Now, on the basis of general physical considerations, we expect that the S-matrix between the massless and massive modes is diagonal and lead to additional phase factors in the Bethe equations.

The aim of this paper is to propose an S-matrix for the massive modes, in order to derive, on a firmer ground, the Bethe equations proposed in Refs. 1 and 2. We shall do this by using an analytic Bethe ansatz involving transfer matrix eigenvalues derived from the diagonalization of $su(1|1)$ and $su(1|1) \times su(1|1)$ invariant S-matrices, respectively for the $AdS_3 \times S^3 \times S^3 \times S^1$ and $AdS_3 \times S^3 \times T^4$ cases. Actually, the symmetries preserved by the vacua of these theories are centrally extended versions of $su(1|1)$ and $su(1|1) \times su(1|1)$, respectively, as argued in Refs. 19 and 20.

In the different context of open AdS_5/CFT_4 spin chains, the analytic Bethe ansatz built on a $su(1|1)$-invariant S-matrix, previously found in Refs. 22 and 23, was already performed, without considering possible scalar factors, by Ref. 24 in order to determine the corresponding transfer matrix eigenvalues and Bethe equations. On the other hand, an $su(1|1) \times su(1|1)$-invariant S-matrix was proposed in Ref. 20 to describe the scattering of magnons in $AdS_3 \times S^3 \times T^4$; however, only magnons in the $su(2)$ sector were analyzed there, in order to derive the dressing phase up to one-loop, and Bethe equations were not derived.

Note Added

Shortly after the first version of this paper was published on the arXiv, another paper[21] appeared. In that paper, two different S-matrices, coming from two different choices of central extension, were derived for $AdS_3 \times S^3 \times S^3 \times S^1$. One can check that the S-matrix written in App. D of Ref. 21 is the same[c] as the S-matrix we have proposed in this paper.

[b]Some first tests of these Bethe equations against string energy calculations have been performed in Ref. 17.

[c]After mapping our variables $x^{\pm}_{1,3}$ to their $x^{\mp}$, $z^{\mp}$, $x^{\pm}_{\bar{1},\bar{3}}$ to $-1/x^{\mp}$, $-1/z^{\mp}$ and $\omega_{1,2}$ to $\eta_{p,q}$. We thank Alessandro Sfondrini for correspondence on this point.

2. Spectrum and S-Matrix

2.1. $\mathrm{AdS}_3 \times S^3 \times T^4$

For the case of $\mathrm{AdS}_3 \times S^3 \times T^4$, the spectrum consists of eight massive modes whose energy–momentum dispersion relation is given by

$$E = \sqrt{1 + 4h^2(\lambda)\sin^2\frac{p}{2}}\,, \tag{2.1}$$

where h is an almost unknown function of the 't Hooft coupling λ: its strong coupling behavior has been predicted to be $h(\lambda) \simeq \sqrt{\lambda}/2\pi$ in Ref. 1, while the one-loop correction has been calculated recently by Ref. 25. These are grouped into bifundamentals of $su(1|1) \times su(1|1)$, which we refer to as "A" and "B". The S-matrices among these bifundamentals are given by tensor products of two $su(1|1)$-invariant S-matrices as follows:

$$S^{(AA)}(p_1,p_2) = S^{(BB)}(p_1,p_2) = S_0(p_1,p_2)[\hat{S}(p_1,p_2) \otimes \hat{S}(p_1,p_2)]\,, \tag{2.2}$$

$$S^{(AB)}(p_1,p_2) = S^{(BA)}(p_1,p_2) = \tilde{S}_0(p_1,p_2)[\hat{S}(p_1,p_2) \otimes \hat{S}(p_1,p_2)]\,, \tag{2.3}$$

where[20,23,24]

$$\hat{S}(p_1,p_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{x_1^- - x_2^-}{x_1^+ - x_2^-} & \dfrac{x_1^+ - x_1^-}{x_1^+ - x_2^-}\dfrac{\omega_2}{\omega_1} & 0 \\ 0 & \dfrac{x_2^+ - x_2^-}{x_1^+ - x_2^-}\dfrac{\omega_1}{\omega_2} & \dfrac{x_1^+ - x_2^+}{x_1^+ - x_2^-} & 0 \\ 0 & 0 & 0 & \dfrac{x_1^- - x_2^+}{x_1^+ - x_2^-} \end{pmatrix} \tag{2.4}$$

and we set $\omega_{1,2} = \omega(p_{1,2}) = 1$. The $x^\pm$ variables are the usual Zhukowsky variables defined by

$$\frac{x^+}{x^-} = e^{ip}\,, \quad x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2i}{h(\lambda)}\,. \tag{2.5}$$

The S-matrix (2.4) satisfies the unitarity condition, but it does not have crossing symmetry.[24] An attempt to derive the crossing symmetry relations for the $\mathfrak{su}(1|1)$ algebra has been put forward in Ref. 20 by using the antipode operation, but this implies, in this case, a transformation on the kinematic variables ($x^\pm \to x^\mp$) that does not correspond to the particle–antiparticle transformation ($x^\pm \to 1/x^\pm$). Then, we guess a possible expression for the scalar factors on the basis of the unitarity and the final matching with the Bethe equations proposed by:[1,2]

$$\begin{aligned} S_0(p_1,p_2) &= \frac{x_1^+ - x_2^-}{x_1^- - x_2^+}\,\frac{1 - \frac{1}{x_1^+ x_2^-}}{1 - \frac{1}{x_1^- x_2^+}}\,\sigma^2(p_1,p_2)\,\frac{x_1^-}{x_1^+}\frac{x_2^+}{x_2^-}\,, \\ \tilde{S}_0(p_1,p_2) &= \sigma^{-2}(p_1,\bar{p}_2)\,\frac{x_1^+}{x_1^-}\frac{x_2^+}{x_2^-}\,, \end{aligned} \tag{2.6}$$

where $\sigma(p_1, p_2)$ was conjectured being the BES dressing phase[26] in Ref. 1[d] and $\bar{p}$ denotes the momentum of an antiparticle, such that $x^{\pm}(\bar{p}) = 1/x^{\pm}(p)$. The scalar factors (2.6) satisfy the relation $S_0(p_1, p_2) = S_0(\bar{p}_1, \bar{p}_2)$, $\tilde{S}_0(p_1, p_2) = \tilde{S}_0(\bar{p}_1, \bar{p}_2)$, that will be important later for the construction of the Bethe equations, and unitarity:

$$S_0(p_1,p_2)S_0(p_2,p_1) = S_{su(2)}(p_1,p_2)\sigma^2(p_1,p_2)\frac{x_1^-}{x_1^+}\frac{x_2^+}{x_2^-} \times S_{su(2)}^{-1}\sigma^{-2}(p_1,p_2)\frac{x_2^-}{x_2^+}\frac{x_1^+}{x_1^-} = 1\,, \tag{2.7}$$

$$\tilde{S}_0(p_1,p_2)\tilde{S}_0(p_2,p_1) = \sigma^{-2}(p_1,\bar{p}_2)\frac{x_1^+}{x_1^-}\frac{x_2^+}{x_2^-}\sigma^{-2}(\bar{p}_2,p_1)\frac{x_1^-}{x_1^+}\frac{x_2^-}{x_2^+} = 1\,, \tag{2.8}$$

where $S_{su(2)}(p_1,p_2) = \frac{x_1^+ - x_2^-}{x_1^- - x_2^+}\frac{1-\frac{1}{x_1^+ x_2^-}}{1-\frac{1}{x_1^- x_2^+}}$.

The Bethe–Yang equations are derived from a periodic boundary condition (PBC). On a circle with circumference L, N_A represents the number of "A" particles with momenta $\{p_1^A, p_2^A, \ldots, p_{N_A}^A\}$ and N_B represents number of "B" particles with momenta $\{p_1^B, p_2^B, \ldots, p_{N_B}^B\}$. Now, we choose an "$A$" particle with a momentum p_j^A and move it around the circle by scattering with all the other particles and similarly for a "B" particle with a momentum p_j^B. Since this virtual process does not change any configuration, we arrive at PBC conditions

$$e^{ip_j^A L} = \prod_{k=1,\neq j}^{N_A} S_0(p_j^A, p_k^A)\prod_{k=1}^{N_B}\tilde{S}_0(p_j^A, p_k^B) \times\left[\hat{T}_{su(1|1)}(p_j^A|\{p_l^A, p_l^B\})\otimes\hat{T}_{su(1|1)}(p_j^A|\{p_l^A, p_l^B\})\right], \tag{2.9}$$

$$e^{ip_j^B L} = \prod_{k=1,\neq j}^{N_B} S_0(p_j^B, p_k^B)\prod_{k=1}^{N_A}\tilde{S}_0(p_j^B, p_k^A) \times\left[\hat{T}_{su(1|1)}(p_j^B|\{p_l^A, p_l^B\})\otimes\hat{T}_{su(1|1)}(p_j^B|\{p_l^A, p_l^B\})\right], \tag{2.10}$$

where $\hat{T}_{su(1|1)}$ is a transfer matrix made of the $su(1|1)$-invariant S-matrix,

$$\hat{T}_{su(1|1)}(p|\{p_l^A\},\{p_l^B\}) = \mathrm{str}_a\left[\hat{S}_{aA_1}(p,p_1^A)\cdots\hat{S}_{aA_{N_A}}(p,p_{N_A}^A)\hat{S}_{aB_1}(p,p_1^B)\cdots\hat{S}_{aB_{N_B}}(p,p_{N_B}^B)\right], \tag{2.11}$$

and a, A_i and B_i stand for a two-dimensional vector space which the S-matrices act on.

[d]Actually, results from Refs. 20 and 27 show that $\sigma(p_1, p_2)$ differs from the BES dressing phase starting from one-loop at strong coupling.

2.2. $\mathbf{AdS_3 \times S^3 \times S^3 \times S^1}$

The spectrum of $\mathrm{AdS}_3 \times S^3 \times S^3 \times S^1$ is a bit more complicated. Denoting l, R_1, R_2 the radii of AdS_3 and the two S^3's respectively, one has the following relation

$$\frac{1}{R_1^2} + \frac{1}{R_2^2} = \frac{1}{l^2} \,. \tag{2.12}$$

By defining $\alpha = l^2/R_1^2$, one can find two massive multiplets, each of which consists of two bosons and two fermions, with two different masses:

$$E_l = \sqrt{m_l^2 + 4h^2(\lambda)\sin^2\frac{p}{2}}\,, \quad l = 1, 3\,, \tag{2.13}$$

where

$$m_1 = \alpha\,, \quad m_3 = 1 - \alpha\,. \tag{2.14}$$

We propose that the four particles with mass m_1 are grouped into two fundamentals of $su(1|1)$, which we refer to "1" and "$\bar{1}$"; and similarly the other four particles with mass m_3 into two additional fundamentals of $su(1|1)$, which we refer to "3" and "$\bar{3}$". In this case, the Zhukowsky variables are defined as:[2]

$$\begin{aligned} x^+_{1,\bar{1}} + \frac{1}{x^+_{1,\bar{1}}} - x^-_{1,\bar{1}} - \frac{1}{x^-_{1,\bar{1}}} &= \frac{2i\alpha}{h(\lambda)}\,, \\ x^+_{3,\bar{3}} + \frac{1}{x^+_{3,\bar{3}}} - x^-_{3,\bar{3}} - \frac{1}{x^-_{3,\bar{3}}} &= \frac{2i(1-\alpha)}{h(\lambda)}\,. \end{aligned} \tag{2.15}$$

The S-matrices among these four doublets are given by single $su(1|1)$-invariant S-matrices as follows:

$$\begin{aligned} S^{(11)}(p_1,p_2) = S^{(33)}(p_1,p_2) &= S^{(\bar{1}\bar{1})}(p_1,p_2) \\ = S^{(\bar{3}\bar{3})}(p_1,p_2) &= S_0(p_1,p_2)\hat{S}(p_1,p_2)\,, \end{aligned} \tag{2.16}$$

$$\begin{aligned} S^{(1\bar{1})}(p_1,p_2) = S^{(3\bar{3})}(p_1,p_2) &= S^{(\bar{1}1)}(p_1,p_2) \\ = S^{(\bar{3}3)}(p_1,p_2) &= \tilde{S}_0(p_1,p_2)\hat{S}(p_1,p_2)\,, \end{aligned} \tag{2.17}$$

$$\begin{aligned} S^{(13)}(p_1,p_2) = S^{(31)}(p_1,p_2) &= S^{(\bar{1}\bar{3})}(p_1,p_2) \\ = S^{(\bar{3}\bar{1})}(p_1,p_2) &= \hat{S}(p_1,p_2)\,, \end{aligned} \tag{2.18}$$

$$\begin{aligned} S^{(1\bar{3})}(p_1,p_2) = S^{(3\bar{1})}(p_1,p_2) &= S^{(\bar{3}1)}(p_1,p_2) \\ = S^{(\bar{1}3)}(p_1,p_2) &= \hat{S}(p_1,p_2)\,, \end{aligned} \tag{2.19}$$

where $\hat{S}(p_1,p_2)$ is given in Eq. (2.4) and the scalar factors S_0 and $\tilde{S}_0$ are defined in Eq. (2.6).

The Bethe–Yang equations can be written in a similar way as before. On a circle with circumference L, we put N_1 number of "1" particles with momenta

$\{p_1^1, p_2^1, \ldots, p_{N_1}^1\}$, $N_{\bar{1}}$ number of "$\bar{1}$" particles with momenta $\{p_1^{\bar{1}}, p_2^{\bar{1}}, \ldots, p_{N_{\bar{1}}}^{\bar{1}}\}$, N_3 number of "3" particles with momenta $\{p_1^3, p_2^3, \ldots, p_{N_3}^3\}$ and $N_{\bar{3}}$ number of "$\bar{3}$" particles with momenta $\{p_1^{\bar{3}}, p_2^{\bar{3}}, \ldots, p_{N_{\bar{3}}}^{\bar{3}}\}$. From these configuration, the PBC equations become

$$e^{ip_j^1 L} = \prod_{k=1,\neq j}^{N_1} S_0(p_j^1, p_k^1) \prod_{k=1}^{N_{\bar{1}}} \tilde{S}_0(p_j^1, p_k^{\bar{1}}) \cdot \hat{T}_{su(1|1)}(p_j^1 | \{p_l^1, p_l^{\bar{1}}, p_l^3, p_l^{\bar{3}}\}) \,, \tag{2.20}$$

$$e^{ip_j^{\bar{1}} L} = \prod_{k=1,\neq j}^{N_{\bar{1}}} S_0(p_j^{\bar{1}}, p_k^{\bar{1}}) \prod_{k=1}^{N_1} \tilde{S}_0(p_j^{\bar{1}}, p_k^1) \cdot \hat{T}_{su(1|1)}(p_j^{\bar{1}} | \{p_l^1, p_l^{\bar{1}}, p_l^3, p_l^{\bar{3}}\}) \,, \tag{2.21}$$

$$e^{ip_j^3 L} = \prod_{k=1,\neq j}^{N_1} S_0(p_j^3, p_k^3) \prod_{k=1}^{N_{\bar{3}}} \tilde{S}_0(p_j^3, p_k^{\bar{3}}) \cdot \hat{T}_{su(1|1)}(p_j^3 | \{p_l^1, p_l^{\bar{1}}, p_l^3, p_l^{\bar{3}}\}) \,, \tag{2.22}$$

$$e^{ip_j^{\bar{3}} L} = \prod_{k=1,\neq j}^{N_{\bar{3}}} S_0(p_j^{\bar{3}}, p_k^{\bar{3}}) \prod_{k=1}^{N_3} \tilde{S}_0(p_j^{\bar{3}}, p_k^3) \cdot \hat{T}_{su(1|1)}(p_j^{\bar{3}} | \{p_l^1, p_l^{\bar{1}}, p_l^3, p_l^{\bar{3}}\}) \,, \tag{2.23}$$

where $\hat{T}_{su(1|1)}$ is given in Eq. (2.11).

3. Derivation of Asymptotic Bethe Ansatz Equations

3.1. *Diagonalization of the transfer matrix*

The $su(1|1)$ transfer matrix has been diagonalized by the analytic Bethe ansatz method in Ref. 24. The eigenvalues can be expressed by

$$\Lambda(p|\{p_\ell\}, \{\lambda_j\}) = \Lambda_0(p|\{p_\ell\}) A(p|\{\lambda_j\}) \,, \tag{3.1}$$

$$\Lambda_0(p|\{p_\ell\}) = 1 - \prod_{\ell=1}^{N} \left(\frac{x^+(p) - x^+(p_\ell)}{x^+(p) - x^-(p_\ell)} \right) , \tag{3.2}$$

$$A(p|\{\lambda_j\}) = \prod_{j=1}^{M} \left(\frac{x^-(p) - x^+(\lambda_j)}{x^+(p) - x^+(\lambda_j)} \right) \tag{3.3}$$

and the magnonic variables λ_j satisfy

$$1 = \prod_{\ell=1}^{N} \left(\frac{x^+(\lambda_j) - x^-(p_\ell)}{x^+(\lambda_j) - x^+(p_\ell)} \right) . \tag{3.4}$$

Here, we have used a short notation that $N = N_A + N_B$ and $\{p_\ell\} = \{p_l^A, p_l^B\}$ for $\mathrm{AdS}_3 \times S^3 \times T^4$; $N = N_1 + N_{\bar{1}} + N_3 + N_{\bar{3}}$, $\{p_\ell\} = \{p_l^1, p_l^{\bar{1}}, p_l^3, p_l^{\bar{3}}\}$ for $\mathrm{AdS}_3 \times S^3 \times S^3 \times S^1$, respectively.

Inserting these into Eqs. (2.9) and (2.10), we get the asymptotic Bethe ansatz equations for AdS$_3 \times S^3 \times T^4$:

$$e^{ip_j^A L} = \prod_{k=1,\neq j}^{N_A} S_0\big(p_j^A, p_k^A\big) \prod_{k=1}^{N_B} \tilde{S}_0(p_j^A, p_k^B) \times \prod_{j=1}^{M} \left(\frac{x^-\big(p_j^A\big) - x^+(\lambda_j)}{x^+\big(p_j^A\big) - x^+(\lambda_j)}\right) \prod_{j=1}^{\bar{M}} \left(\frac{x^-\big(p_j^A\big) - x^+(\bar{\lambda}_j)}{x^+\big(p_j^A\big) - x^+(\bar{\lambda}_j)}\right), \tag{3.5}$$

$$e^{ip_j^B L} = \prod_{k=1,\neq j}^{N_B} S_0\big(p_j^B, p_k^B\big) \prod_{k=1}^{N_A} \tilde{S}_0\big(p_j^B, p_k^A\big) \times \prod_{j=1}^{M} \left(\frac{x^-\big(p_j^B\big) - x^+(\lambda_j)}{x^+\big(p_j^B\big) - x^+(\lambda_j)}\right) \prod_{j=1}^{\bar{M}} \left(\frac{x^-\big(p_j^B\big) - x^+(\bar{\lambda}_j)}{x^+\big(p_j^B\big) - x^+(\bar{\lambda}_j)}\right), \tag{3.6}$$

$$1 = \prod_{l=1}^{N_A} \left(\frac{x^+(\lambda_j) - x^-\big(p_l^A\big)}{x^+(\lambda_j) - x^+\big(p_l^A\big)}\right) \prod_{l=1}^{N_B} \left(\frac{x^+(\lambda_j) - x^-\big(p_l^B\big)}{x^+(\lambda_j) - x^+\big(p_l^B\big)}\right), \tag{3.7}$$

$$1 = \prod_{l=1}^{N_A} \left(\frac{x^+(\bar{\lambda}_j) - x^-\big(p_l^A\big)}{x^+(\bar{\lambda}_j) - x^+\big(p_l^A\big)}\right) \prod_{l=1}^{N_B} \left(\frac{x^+(\bar{\lambda}_j) - x^-\big(p_l^B\big)}{x^+(\bar{\lambda}_j) - x^+\big(p_l^B\big)}\right). \tag{3.8}$$

This can be represented pictorially as in Fig. 1.

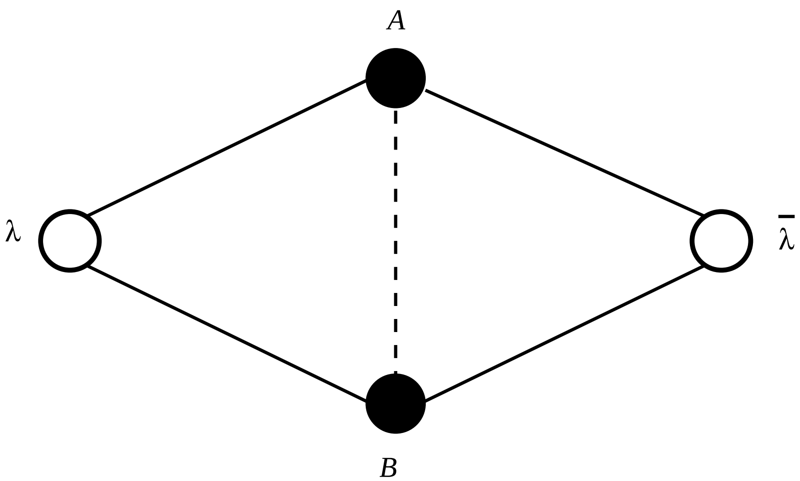

Fig. 1. AdS$_3 \times S^3 \times T^4$: two momentum-carrying nodes (black dots) are connected to two magnonic nodes (circle).

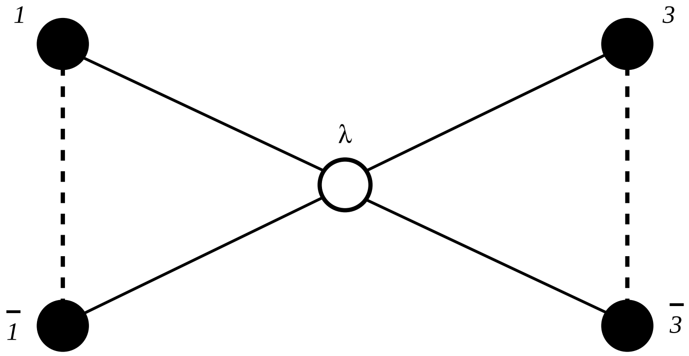

Fig. 2. $AdS_3 \times S^3 \times S^3 \times S^1$: four momentum-carrying nodes (black dots) are connected to a single magnonic node (circle).

Similarly, from Eqs. (2.20)–(2.23), we obtain the asymptotic Bethe ansatz equations for $AdS_3 \times S^3 \times S^3 \times S^1$:

$$e^{ip_j^1 L} = \prod_{k=1,\neq j}^{N_1} S_0\big(p_j^1, p_k^1\big) \prod_{k=1}^{N_{\bar{1}}} \tilde{S}_0\big(p_j^1, p_k^{\bar{1}}\big) \prod_{j=1}^{M} \left(\frac{x^-\big(p_j^1\big) - x^+(\lambda_j)}{x^+\big(p_j^1\big) - x^+(\lambda_j)} \right), \tag{3.9}$$

$$e^{ip_j^{\bar{1}} L} = \prod_{k=1,\neq j}^{N_{\bar{1}}} S_0\big(p_j^{\bar{1}}, p_k^{\bar{1}}\big) \prod_{k=1}^{N_1} \tilde{S}_0\big(p_j^{\bar{1}}, p_k^1\big) \prod_{j=1}^{M} \left(\frac{x^-\big(p_j^{\bar{1}}\big) - x^+(\lambda_j)}{x^+\big(p_j^{\bar{1}}\big) - x^+(\lambda_j)} \right), \tag{3.10}$$

$$e^{ip_j^3 L} = \prod_{k=1,\neq j}^{N_1} S_0\big(p_j^3, p_k^3\big) \prod_{k=1}^{N_{\bar{3}}} \tilde{S}_0\big(p_j^3, p_k^{\bar{3}}\big) \prod_{j=1}^{M} \left(\frac{x^-\big(p_j^3\big) - x^+(\lambda_j)}{x^+\big(p_j^3\big) - x^+(\lambda_j)} \right), \tag{3.11}$$

$$e^{ip_j^{\bar{3}} L} = \prod_{k=1,\neq j}^{N_{\bar{3}}} S_0\big(p_j^{\bar{3}}, p_k^{\bar{3}}\big) \prod_{k=1}^{N_3} \tilde{S}_0\big(p_j^{\bar{3}}, p_k^3\big) \prod_{j=1}^{M} \left(\frac{x^-\big(p_j^{\bar{3}}\big) - x^+(\lambda_j)}{x^+\big(p_j^{\bar{3}}\big) - x^+(\lambda_j)} \right), \tag{3.12}$$

$$1 = \prod_{\ell=1}^{N} \left(\frac{x^+(\lambda_j) - x^-(p_\ell)}{x^+(\lambda_j) - x^+(p_\ell)} \right). \tag{3.13}$$

These sets of Bethe ansatz equations can be represented pictorially as in Fig. 2.

3.2. *Comparison to the Bethe equations of Refs.* 1 *and* 2

In order to translate Eqs. (3.5)–(3.8) into the notation of Refs. 1 and 2, we have to replace p_B by $\bar{p}_B$ and disentangle the two "magnonic" variables into four. In Eq. (3.5), for instance, the first step involves the second factor:

$$\prod_{k=1}^{N_B} \tilde{S}_0\big(p_j^A, \bar{p}_k^B\big) = \prod_{k=1}^{N_B} \sigma^{-2}\big(p_j^A, p_k^B\big) \frac{x_j^+}{x_j^-} \frac{x_k^-}{x_k^+}\,. \tag{3.14}$$

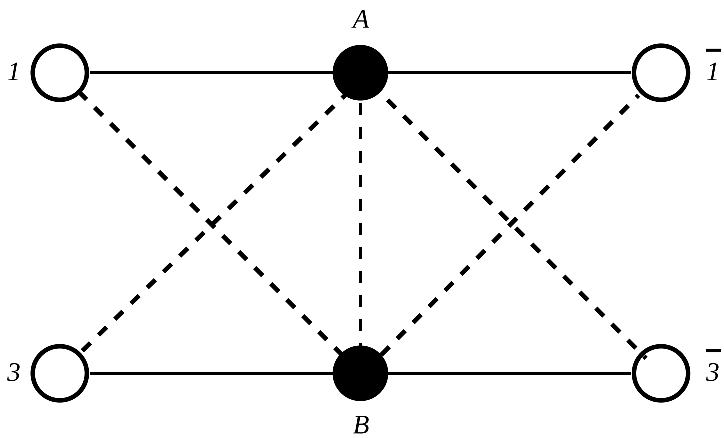

Fig. 3. $AdS_3 \times S^3 \times T^4$: two momentum-carrying nodes (black dots) are connected to four magnonic nodes (circle) after redefinition.

Now, since from the momentum constraint we have that $\prod_{k=1}^{N_A} \frac{x_k^+}{x_k^-} \prod_{k=1}^{N_B} \frac{x_k^-}{x_k^+} = 1$ and $\frac{x_j^+}{x_j^-} = e^{ip_j}$, finally we get, ignoring for the moment the magnonic part (setting to zero both M and $\bar{M}$):

$$e^{ip_j^A(L+N_A-N_B)} = \prod_{k=1,\neq j}^{N_A} \frac{x_j^+ - x_k^-}{x_j^- - x_k^+} \frac{1 - \frac{1}{x_j^+ x_k^-}}{1 - \frac{1}{x_j^- x_k^+}} \sigma^2\big(p_j^A, p_k^A\big) \prod_{k=1}^{N_B} \sigma^{-2}\big(p_j^A, p_k^B\big)\,. \tag{3.15}$$

In the case of Eq. (3.6), we get:

$$e^{-ip_j^B(L+N_B-N_A)} = \prod_{k=1,\neq j}^{N_B} \frac{x_j^+ - x_k^-}{x_j^- - x_k^+} \frac{1 - \frac{1}{x_j^+ x_k^-}}{1 - \frac{1}{x_j^- x_k^+}} \sigma^2\big(p_j^B, p_k^B\big) \prod_{k=1}^{N_A} \sigma^{-2}\big(p_j^B, p_k^A\big)\,. \tag{3.16}$$

Now, in order to complete the comparison, we need also to redefine the magnonic variables (after this, Fig. 1 changes to Fig. 3):

$$\begin{aligned} x^+(\lambda_j) &= x_{1,j}\,, \quad j = 1, \ldots, K_1\,; \\ x^+(\lambda_{K_1+j}) &= \frac{1}{x_{\bar{1},j}}\,, \quad j = 1, \ldots, K_{\bar{1}}\,, \quad M = K_1 + K_{\bar{1}}\,, \end{aligned} \tag{3.17}$$

$$\begin{aligned} x^+(\bar{\lambda}_j) &= x_{3,j}\,, \quad j = 1, \ldots, K_3\,; \\ x^+(\bar{\lambda}_{K_3+j}) &= \frac{1}{x_{\bar{3},j}}\,, \quad j = 1, \ldots, K_3\,, \quad M = K_3 + K_{\bar{3}}\,. \end{aligned} \tag{3.18}$$

Then, Eqs. (3.5)–(3.8) become:

$$e^{ip_j^A(L+K_A-K_B+K_{\bar{1}}+K_{\bar{3}})}$$
$$= \prod_{k=1,\neq j}^{K_A} \frac{x_j^+ - x_k^-}{x_j^- - x_k^+} \frac{1-\frac{1}{x_j^+ x_k^-}}{1-\frac{1}{x_j^- x_k^+}} \sigma^2\left(p_j^A, p_k^A\right) \prod_{k=1}^{N_B} \sigma^{-2}\left(p_j^A, p_k^B\right)$$
$$\times \prod_{j=1}^{K_1} \frac{x^-\left(p_j^A\right) - x_{1,j}}{x^+\left(p_j^A\right) - x_{1,j}} \prod_{j=1}^{K_{\bar{1}}} \frac{1-\frac{1}{x^-(p_j^A)x_{\bar{1},j}}}{1-\frac{1}{x^+(p_j^A)x_{\bar{1},j}}}$$
$$\times \prod_{j=1}^{K_3} \frac{x^-\left(p_j^A\right) - x_{3,j}}{x^+\left(p_j^A\right) - x_{3,j}} \prod_{j=1}^{K_{\bar{3}}} \frac{1-\frac{1}{x^-(p_j^A)x_{\bar{3},j}}}{1-\frac{1}{x^+(p_j^A)x_{\bar{3},j}}}, \tag{3.19}$$

$$e^{-ip_j^B(L+K_A-K_B+K_{\bar{1}}+K_{\bar{3}})}$$
$$= \prod_{k=1,\neq j}^{K_B} \frac{x_j^+ - x_k^-}{x_j^- - x_k^+} \frac{1-\frac{1}{x_j^+ x_k^-}}{1-\frac{1}{x_j^- x_k^+}} \sigma^2\left(p_j^B, p_k^B\right) \prod_{k=1}^{N_A} \sigma^{-2}\left(p_j^B, p_k^A\right)$$
$$\times \prod_{j=1}^{K_{\bar{1}}} \frac{x^-\left(p_j^B\right) - x_{\bar{1},j}}{x^+\left(p_j^B\right) - x_{\bar{1},j}} \prod_{j=1}^{K_1} \frac{1-\frac{1}{x^-(p_j^B)x_{1,j}}}{1-\frac{1}{x^+(p_j^B)x_{1,j}}}$$
$$\times \prod_{j=1}^{K_{\bar{3}}} \frac{x^-\left(p_j^B\right) - x_{\bar{3},j}}{x^+\left(p_j^B\right) - x_{\bar{3},j}} \prod_{j=1}^{K_3} \frac{1-\frac{1}{x^-(p_j^B)x_{3,j}}}{1-\frac{1}{x^+(p_j^B)x_{3,j}}}, \tag{3.20}$$

$$1 = \prod_{l=1}^{N_A} \left(\frac{x_{1,j} - x^-\left(p_l^A\right)}{x_{1,j}^+ - x^+\left(p_l^A\right)} \right) \prod_{l=1}^{N_B} \left(\frac{1-\frac{1}{x_{1,j}x^-(p_l^B)}}{1-\frac{1}{x_{1,j}x^+(p_l^B)}} \right), \tag{3.21}$$

$$1 = \prod_{l=1}^{N_A} \left(\frac{x_{3,j} - x^-\left(p_l^A\right)}{x_{3,j}^+ - x^+\left(p_l^A\right)} \right) \prod_{l=1}^{N_B} \left(\frac{1-\frac{1}{x_{3,j}x^-(p_l^B)}}{1-\frac{1}{x_{3,j}x^+(p_l^B)}} \right), \tag{3.22}$$

$$1 = \prod_{l=1}^{N_B} \left(\frac{x_{\bar{1},j} - x^-\left(p_l^B\right)}{x_{\bar{1},j}^+ - x^+\left(p_l^B\right)} \right) \prod_{l=1}^{N_A} \left(\frac{1-\frac{1}{x_{\bar{1},j}x^-(p_l^A)}}{1-\frac{1}{x_{\bar{1},j}x^+(p_l^A)}} \right), \tag{3.23}$$

$$1 = \prod_{l=1}^{N_B} \left(\frac{x_{\bar{3},j} - x^-\left(p_l^B\right)}{x_{\bar{3},j}^+ - x^+\left(p_l^B\right)} \right) \prod_{l=1}^{N_A} \left(\frac{1-\frac{1}{x_{\bar{3},j}x^-(p_l^A)}}{1-\frac{1}{x_{\bar{3},j}x^+(p_l^A)}} \right), \tag{3.24}$$

which match exactly the equations conjectured by Refs. 1 and 2, if we define $L_{[1,2]} = L + K_A - K_B + K_{\bar{1}} + K_{\bar{3}})$.

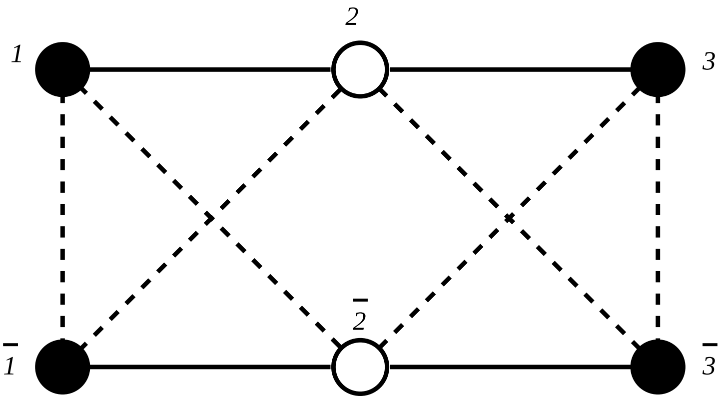

Fig. 4. $\mathrm{AdS}_3 \times S^3 \times S^3 \times S^1$: four momentum-carrying nodes (black dots) are connected to two magnonic nodes (circle) after redefinition.

In analogy with the case of $\mathrm{AdS}_3 \times T^4$, in order to get the Bethe equations written in the notation of Refs. 1 and 2, we need to change $p^{\bar{1},\bar{3}} \to \bar{p}^{\bar{1},\bar{3}}$ in Eqs. (3.9)–(3.13) and to redefine the magnonic nodes corresponding to the variable 2 into two sets of 2 and $\bar{2}$ variables,

$$\begin{aligned} x^+(\lambda_j) &= x_{2,j}\,, \quad j = 1, \ldots, K_2\,; \\ x^+(\lambda_{j+K_2}) &= \frac{1}{x_{\bar{2},j}}\,, \quad j = 1, \ldots, K_{\bar{2}}\,, \end{aligned} \tag{3.25}$$

as illustrated in Fig. 4:

$$\begin{aligned} & e^{ip_j^1(L+N_1-N_{\bar{1}}+K_{\bar{2}})} \\ &\quad = e^{i(P_1-P_{\bar{1}})} \prod_{k=1,\neq j}^{N_1} \frac{x_{1,j}^+ - x_{1,k}^-}{x_{1,j}^- - x_{1,k}^+} \frac{1 - \frac{1}{x_{1,j}^+ x_{1,k}^-}}{1 - \frac{1}{x_{1,j}^- x_{1,k}^+}} \sigma^2(p_{1,j}, p_{1,k}) \prod_{k=1}^{N_{\bar{1}}} \sigma^{-2}(p_{1,j}, p_{\bar{1},k}) \\ &\qquad \times \prod_{j=1}^{K_2} \left(\frac{x^-(p_j^1) - x_{2,j}}{x^+(p_j^1) - x_{2,j}} \right) \prod_{j=1}^{K_{\bar{2}}} \left(\frac{1 - \frac{1}{x^-(p_j^1) x_{2,j}}}{1 - \frac{1}{x^-(p_j^1) x_{2,j}}} \right), \end{aligned} \tag{3.26}$$

$$\begin{aligned} & e^{ip_j^3(L+N_3-N_{\bar{3}}+K_{\bar{2}})} \\ &\quad = e^{i(P_3-P_{\bar{3}})} \prod_{k=1,\neq j}^{N_3} \frac{x_{3,j}^+ - x_{3,k}^-}{x_{3,j}^- - x_{3,k}^+} \frac{1 - \frac{1}{x_{3,j}^+ x_{3,k}^-}}{1 - \frac{1}{x_{3,j}^- x_{3,k}^+}} \sigma^2(p_{3,j}, p_{3,k}) \prod_{k=1}^{N_{\bar{3}}} \sigma^{-2}(p_{3,j}, p_{\bar{3},k}) \\ &\qquad \times \prod_{j=1}^{K_2} \left(\frac{x^-(p_j^3) - x_{2,j}}{x^+(p_j^3) - x_{2,j}} \right) \prod_{j=1}^{K_{\bar{2}}} \left(\frac{1 - \frac{1}{x^-(p_j^3) x_{2,j}}}{1 - \frac{1}{x^-(p_j^3) x_{2,j}}} \right), \end{aligned} \tag{3.27}$$

$$
\begin{aligned}
& e^{-ip_j^{\bar{1}}(L+N_{\bar{1}}-N_1+K_2)} \\
& \quad = e^{i(P_{\bar{1}}-P_1)} \prod_{k=1,\neq j}^{N_{\bar{1}}} \frac{x_{\bar{1},j}^{+}-x_{\bar{1},k}^{-}}{x_{\bar{1},j}^{-}-x_{\bar{1},k}^{+}} \frac{1-\frac{1}{x_{\bar{1},j}^{+}x_{\bar{1},k}^{-}}}{1-\frac{1}{x_{\bar{1},j}^{-}x_{\bar{1},k}^{+}}} \sigma^2(p_{\bar{1},j},p_{\bar{1},k}) \prod_{k=1}^{N_1} \sigma^{-2}(p_{\bar{1},j},p_{1,k}) \\
& \qquad \times \prod_{j=1}^{K_{\bar{2}}} \left(\frac{x^{-}\left(p_j^{\bar{1}}\right)-x_{\bar{2},j}}{x^{+}\left(p_j^{\bar{1}}\right)-x_{\bar{2},j}} \right) \prod_{j=1}^{K_2} \left(\frac{1-\frac{1}{x^{-}(p_j^{\bar{1}})x_{2,j}}}{1-\frac{1}{x^{-}(p_j^{\bar{1}})x_{2,j}}} \right) ,
\end{aligned}
\tag{3.28}
$$

$$
\begin{aligned}
& e^{-ip_j^{\bar{3}}(L+N_{\bar{3}}-N_3+K_2)} \\
& \quad = e^{i(P_{\bar{3}}-P_3)} \prod_{k=1,\neq j}^{N_{\bar{3}}} \frac{x_{\bar{3},j}^{+}-x_{\bar{3},k}^{-}}{x_{\bar{3},j}^{-}-x_{\bar{3},k}^{+}} \frac{1-\frac{1}{x_{\bar{3},j}^{+}x_{\bar{3},k}^{-}}}{1-\frac{1}{x_{\bar{3},j}^{-}x_{\bar{3},k}^{+}}} \sigma^2(p_{\bar{3},j},p_{\bar{3},k}) \prod_{k=1}^{N_3} \sigma^{-2}(p_{\bar{3},j},p_{3,k}) \\
& \qquad \times \prod_{j=1}^{K_{\bar{2}}} \left(\frac{x^{-}\left(p_j^{\bar{3}}\right)-x_{\bar{2},j}}{x^{+}\left(p_j^{\bar{3}}\right)-x_{\bar{2},j}} \right) \prod_{j=1}^{K_2} \left(\frac{1-\frac{1}{x^{-}(p_j^{\bar{3}})x_{2,j}}}{1-\frac{1}{x^{-}(p_j^{\bar{3}})x_{2,j}}} \right) ,
\end{aligned}
\tag{3.29}
$$

$$
\begin{aligned}
1 = & \prod_{\ell=1}^{K_1} \frac{x_{2,j}-x^{-}(p_{1,\ell})}{x_{2,j}-x^{+}(p_{1,\ell})} \prod_{\ell=1}^{K_3} \frac{x_{2,j}-x^{-}(p_{3,\ell})}{x_{2,j}-x^{+}(p_{3,\ell})} \\
& \times \prod_{\ell=1}^{K_{\bar{1}}} \frac{1-\frac{1}{x_{2,j}x^{-}(p_{\bar{1},\ell})}}{1-\frac{1}{x_{2,j}x^{+}(p_{\bar{1},\ell})}} \prod_{\ell=1}^{K_{\bar{3}}} \frac{1-\frac{1}{x_{2,j}x^{-}(p_{\bar{3},\ell})}}{1-\frac{1}{x_{2,j}x^{+}(p_{\bar{3},\ell})}} ,
\end{aligned}
\tag{3.30}
$$

$$
\begin{aligned}
1 = & \prod_{\ell=1}^{K_{\bar{1}}} \frac{x_{\bar{2},j}-x^{-}(p_{\bar{1},\ell})}{x_{\bar{2},j}-x^{+}(p_{\bar{1},\ell})} \prod_{\ell=1}^{K_{\bar{3}}} \frac{x_{\bar{2},j}-x^{-}(p_{\bar{3},\ell})}{x_{\bar{2},j}-x^{+}(p_{\bar{3},\ell})} \\
& \times \prod_{\ell=1}^{K_1} \frac{1-\frac{1}{x_{\bar{2},j}x^{-}(p_{1,\ell})}}{1-\frac{1}{x_{\bar{2},j}x^{+}(p_{1,\ell})}} \prod_{\ell=1}^{K_3} \frac{1-\frac{1}{x_{\bar{2},j}x^{-}(p_{3,\ell})}}{1-\frac{1}{x_{\bar{2},j}x^{+}(p_{3,\ell})}} .
\end{aligned}
\tag{3.31}
$$

In this case, to have full agreement with Refs. 1 and 2, we have to redefine the parameter L in different ways in each equation for the momentum-carrying variables:

$$
L_1 \equiv L + N_1 - N_{\bar{1}} + K_{\bar{2}} , \quad L_3 \equiv L + N_3 - N_{\bar{3}} + K_{\bar{2}} ,
$$

$$
L_{\bar{1}} \equiv L + N_{\bar{1}} - N_1 + K_{\bar{2}} , \quad L_{\bar{3}} \equiv L + N_{\bar{3}} - N_3 + K_{\bar{2}} .
$$

Independent definitions of the spin chain lengths could be not so strange, since it has been already needed in Ref. 17 in order to solve an apparent disagreement between string results and predictions from the Bethe equations for energies of solutions belonging to the $su(2) \times su(2)$ sector.

4. Discussion

We proposed $su(1|1) \times su(1|1)$- and $su(1|1)$-invariant S-matrices for the massive modes of IIB string theory on $\mathrm{AdS}_3 \times S^3 \times T^4$ and $\mathrm{AdS}_3 \times S^3 \times S^3 \times S^1$, respectively. From these we derived the Bethe equations proposed in Refs. 1 and 2. The derivation involved, among other steps, the particle–antiparticle transformation on some momenta of the "massive" variables and the doubling of the fermionic variables in a fashion similar to the $\mathrm{AdS}_5\mathrm{CFT}_4$ (Ref. 28) and $\mathrm{AdS}_4/\mathrm{CFT}_3$ (Ref. 29) cases.

Some scalar factors remained undetermined in our proposal and we were able to guess them by requiring the unitarity of the S-matrix and the matching with the conjectured BAEs. Because of the apparently missing crossing relations[30] for the $\mathfrak{su}(1|1)$ algebra,[20,24] a more solid derivation of such scalar factors remains as an open problem.

Another open problem is to incorporate the massless modes into the S-matrix formulation. In a relativistic theory, the massless limit can be obtained by shifting the rapidity to $\pm\infty$, which often makes the S-matrices between massive and massless modes trivial. While this mechanism seems unapplicable in our nonrelativistic case, we believe a similar argument may provide a clue.

Albeit these unsolved problems, we believe that our findings can lead to some deeper understanding of the yet quite unexplored $\mathrm{AdS}_3/\mathrm{CFT}_2$.

Finally, it would be interesting to investigate possible exact S-matrices for the analogous case of $\mathrm{AdS}_2/\mathrm{CFT}_1$, for which a set of all-loop Bethe equations has been recently proposed in Ref. 31.[e] It will also be interesting to check our proposals in certain perturbative computations. One immediate way is to compute the worldsheet S-matrix based on a gauge-fixed string action for the strong coupling limit.[f] On the other hand, it would also be important to check the reflectionless of our S-matrices, as predicted by Ref. 20, through some weak coupling perturbative calculations, for example, or along the lines of Refs. 34–36.

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[e]See also Ref. 32 for the derivation of the classical equations in this case and a review about finite-gap integration in various AdS_d backgrounds.

[f]Our proposal seems to be not compatible with the worldsheet results of Ref. 33. In that paper some scattering elements at strong coupling have been calculated, however a full computation in that regime, as well as weak coupling results coming from a spin chain formulation, would be desirable before excluding our proposal completely.

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