Integrability in AdS/CFT

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Plan

- Lecture 1. Perturbative integrability
- Lecture 2. Nonperturbative integrability: S-matrix
- Lecture 3. Finite-size effects

Ref: N. Beisert et.al. “Review of AdS/CFT Integrability” arXiv:1012.3982-4005

Lecture 1. Perturbative integrability

Plan

1. Introduction to AdS/CFT

2. Perturbative integrability in N=4 SYM

3. Classical integrability in string theory on $AdS_5 \times S^5$

AdS / CFT duality

- Type IIB superstrings on $AdS_5 \times S^5$

dual to

$\mathcal{N}=4 \quad SU(N_c)$ super-Yang-Mills theory

[Maldacena (1997)]

AdS / CFT duality

- Parameter relations:

$$\boxed{g_s = \frac{4\pi\lambda}{N_c} \quad \& \quad \frac{R^2}{\alpha'} = \sqrt{\lambda}}$$

with ‘t Hooft coupling $\lambda = N_c g_{\mathrm{YM}}^2$

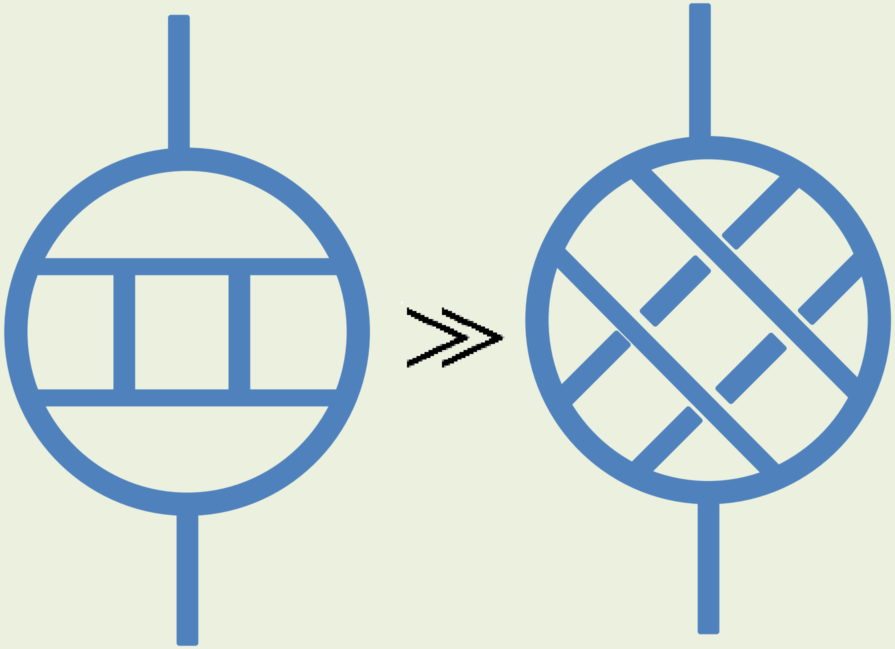
- Free superstring theory corresponds to a planar limit of SYM

$$g_s \to 0 \;\equiv\; N_c \to \infty \text{ with fixed } \lambda$$

- Quantitative check is tricky since it is a strong-weak duality
  - SYM perturbation for $\lambda \ll 1$
  - String perturbation for $\alpha' \ll 1 \quad \Rightarrow \quad \lambda \gg 1$

Planar Limit

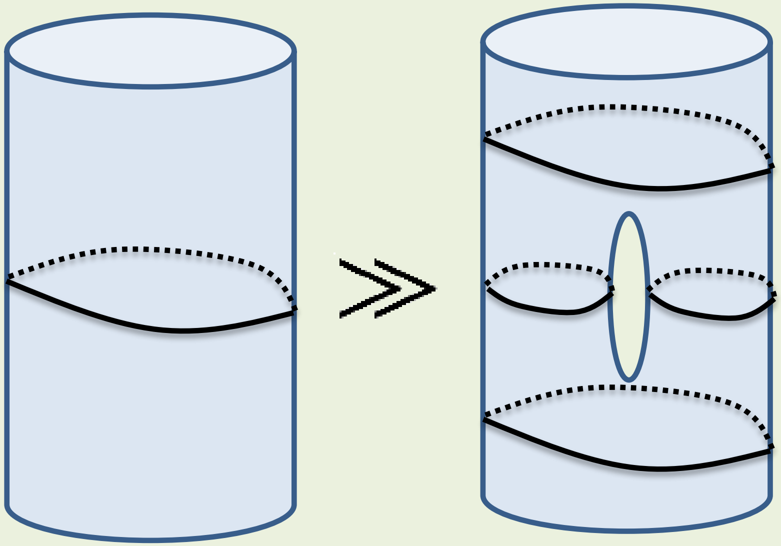
$$N_c \to \infty$$



$g_{\mathrm{YM}}^{10} N_c^5 = \lambda^5$

$g_{\mathrm{YM}}^{10} N_c^3 = \dfrac{\lambda^5}{N_c^2}$

$$g_s \to 0$$



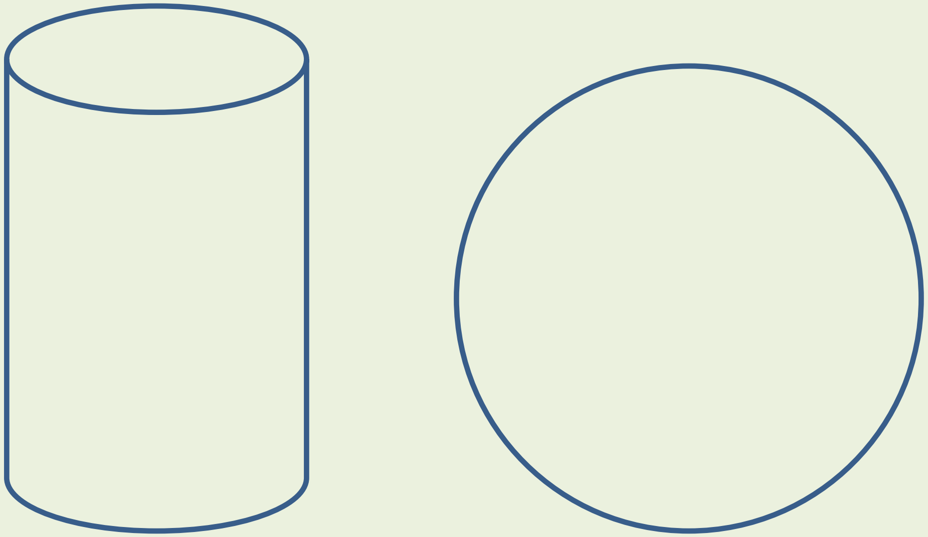
$O(g_s^2)$

SYM Operator vs. string configuration

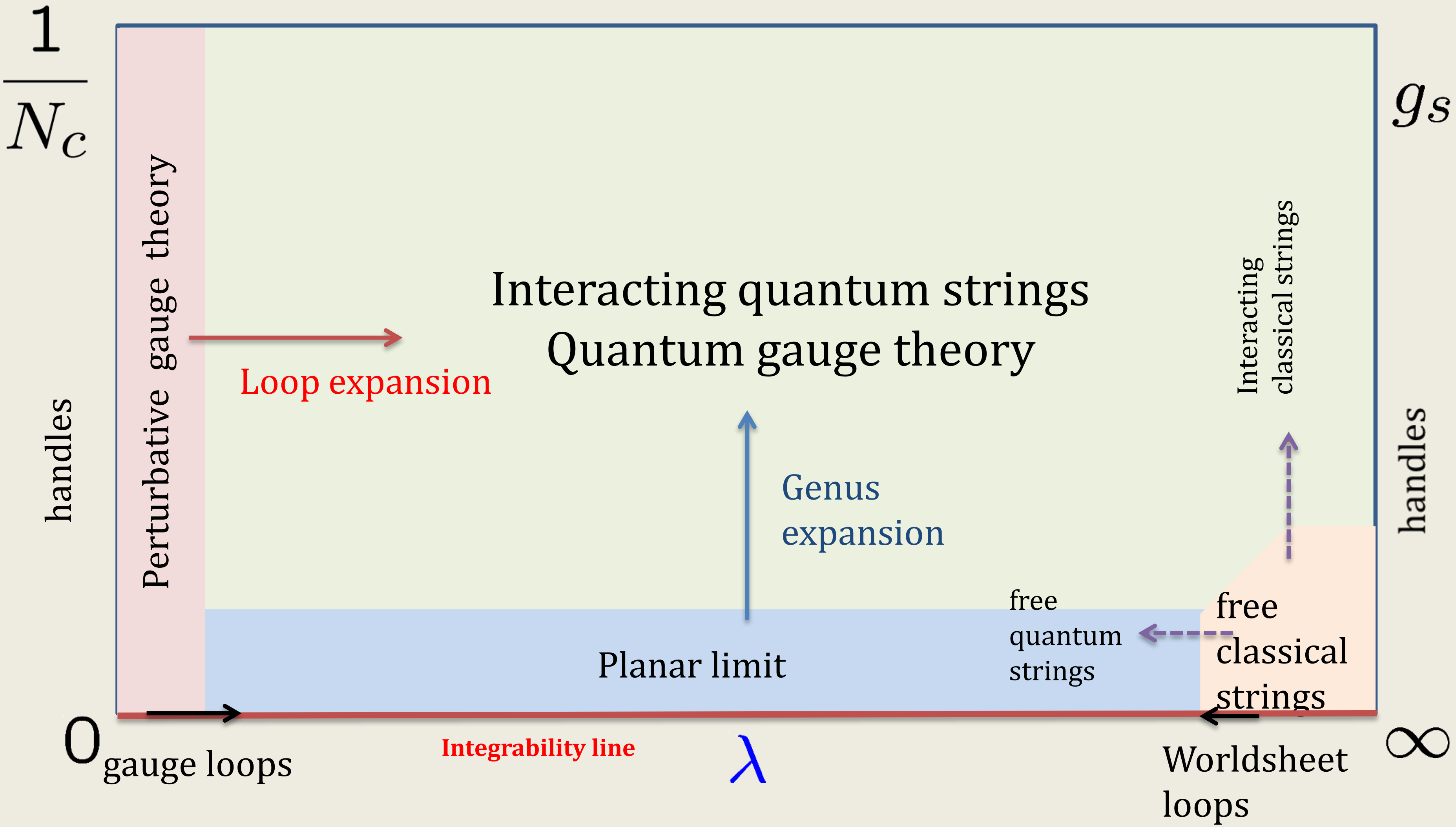
- Composite SYM operator

$$\mathcal{O}(x) = \mathrm{Tr}\left[XYZF_{\mu\nu}\chi^{\alpha}(D_{\mu}Y)\ldots\right]$$

- String configuration in a target space



Parameter space



$\frac{1}{N_c}$

g_s

handles

Perturbative gauge theory

Loop expansion

Interacting quantum strings

Quantum gauge theory

Interacting classical strings

handles

Genus expansion

Planar limit

free quantum strings

free classical strings

0 gauge loops

Integrability line

λ

Worldsheet loops

∞

Integrability

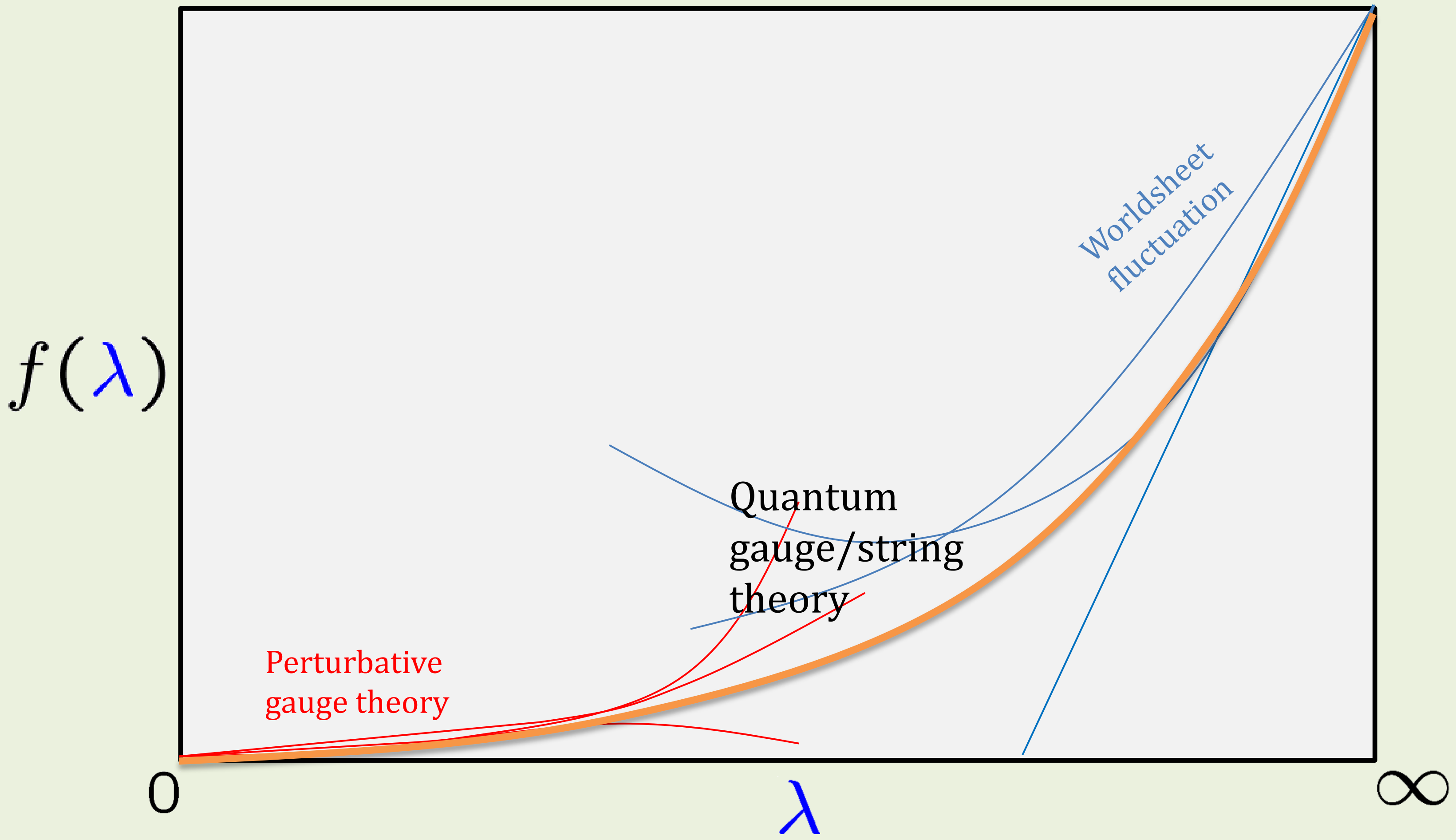
- Appears in the planar limit

- Perturbative integrability
  - Certain integrable models appear in perturbative computations
  - Classical string solutions from some classical integrable systems

- Nonperturbative integrability
  - Exact results for any value of λ

- Only a few physical quantities are exactly computable so far
  - Anomalous dimensions
  - Worldsheet S-matrix

Nonperturbative



Perturbative integrability in N=4 SYM

N=4 Super Yang-Mills theory

•$\mathcal{N}=4\ \ SU(N_c)$ SYM

$$\boxed{S=\frac{\mathrm{Tr}}{g_{\mathrm{YM}}^2}\int d^4x\left\{-\frac{1}{4}F_{\mu\nu}^2+(D_\mu\Phi^a)^2+\left[\Phi^a,\Phi^b\right]^2+\bar{\chi}\not{D}\chi-i\bar{\chi}\Gamma_a[\Phi^a,\chi]\right\}}$$

- R-symmetry : *N*=4 SUSY so(6) ≅ su(4)
- Scalar fields : $\Phi^a,\ a=1,\ldots,6$ $\begin{array}{|c|}\hline \\\hline \\\hline\end{array}$
- Gauginos :$\chi,\ \bar{\chi}$ fundamental in su(4) $\square$
- All in adjoint rep. in $SU(N_c)$

	R-charge
A_μ	$\mathbf{1}$
χ_α^A $\bar{\chi}_{\bar{\alpha}}^{\bar{A}}$	$\mathbf{4}\oplus\bar{\mathbf{4}}$
Φ^a	$\mathbf{6}$

4d conformal field theory

- One-loop β-function

$$\beta \equiv \mu \frac{\partial g_{\mathrm{YM}}}{\partial \mu} = -\frac{g_{\mathrm{YM}}^3}{16\pi^2}\left(\frac{11}{3}N_c - \frac{1}{6}\overset{6N_c}{\sum_i C_i} - \frac{1}{3}\overset{8N_c}{\sum_j \tilde{C}_j}\right) \quad = 0$$

- β =0 at all orders of perturbation
  - Three loops in superspace formulation
  - All loops in light-cone gauge

- No scale dependence

N=4 superconformal algebra

- Lorentz generators : $L_{\mu\nu}$
- Translations : P_μ
- Conformal boosts : K_μ
- Dilatation : D

su(2,2) ≅ so(2,4)

- Supercharges : $Q_{a\alpha}, \quad \bar{Q}^{a}_{\dot{\alpha}}$
- Superconformal boosts : $S^{a}_{\alpha}, \quad \bar{S}_{\dot{\alpha}a}$

32 super charges

- R-symmetry : su(4)

psu(2,2|4)

$$\left(\begin{array}{c|c|c} L & Q & P \\ \hline S & R & \bar{Q} \\ \hline K & \bar{S} & \bar{L} \end{array}\right)$$

- psu(2,2|4) commutation relations

$$\boxed{\begin{array}{l} \left[D, P_\mu\right] = -iP_\mu, \qquad \left[D, L_{\mu\nu}\right] = 0, \qquad \left[D, K_\mu\right] = iK_\mu \\ [D, Q_{\alpha a}] = -\frac{i}{2} Q_{\alpha a}, \quad \left[D, \overline{Q}^{a}_{\dot\alpha}\right] = -\frac{i}{2}\overline{Q}^{a}_{\dot\alpha}, \quad [D, S^{\ a}_{\alpha}] = \frac{i}{2} S^{\ a}_{\alpha}, \quad \left[D, \overline{S}_{\dot\alpha a}\right] = \frac{i}{2}\overline{S}_{\dot\alpha a} \end{array}}$$

$$\left[L_{\mu\nu}, P_\lambda\right] = -i(\eta_{\mu\lambda} P_\nu - \eta_{\lambda\nu} P_\mu), \qquad \left[L_{\mu\nu}, K_\lambda\right] = -i(\eta_{\mu\lambda} K_\nu - \eta_{\lambda\nu} K_\mu)$$

$$\left[P_\mu, K_\nu\right] = 2i(L_{\mu\nu} - \eta_{\mu\nu} D)$$

$$\left\{Q_{\alpha a}, \overline{Q}^{b}_{\dot\alpha}\right\} = \gamma^{\mu}_{\alpha\dot\alpha}\delta^{b}_{a} P_\mu, \quad \{Q_{\alpha a}, Q_{\alpha b}\} = \left\{\overline{Q}^{a}_{\dot\alpha}, \overline{Q}^{b}_{\dot\alpha}\right\} = 0$$

$$\left[P_\mu, Q_{\alpha a}\right] = \left[P_\mu, \overline{Q}^{a}_{\dot\alpha}\right] = 0, \quad [L^{\mu\nu}, Q_{\alpha a}] = i\gamma^{\mu\nu}_{\alpha\beta}\epsilon^{\beta\gamma} Q_{\gamma a}, \quad \left[L^{\mu\nu}, \overline{Q}^{a}_{\dot\alpha}\right] = i\gamma^{\mu\nu}_{\dot\alpha\dot\beta}\epsilon^{\dot\beta\dot\gamma}\overline{Q}^{a}_{\dot\gamma}$$

$$[K^\mu, Q_{\alpha a}] = \gamma^{\mu}_{\alpha\dot\alpha}\epsilon^{\dot\alpha\dot\beta}\overline{S}_{\dot\beta a}, \quad \left[K^\mu, \overline{Q}^{a}_{\dot\alpha}\right] = \gamma^{\mu}_{\alpha\dot\alpha}\epsilon^{\alpha\beta} S^{\ a}_{\beta}$$

$$\left\{Q_{\alpha a}, \overline{Q}^{b}_{\dot\alpha}\right\} = \gamma^{\mu}_{\alpha\dot\alpha}\delta^{b}_{a} P_\mu, \quad \left\{S^{\ a}_{\alpha}, \overline{S}_{\dot\alpha b}\right\} = \gamma^{\mu}_{\alpha\dot\alpha}\delta^{a}_{b} K_\mu, \quad \{S^{\ a}_{\alpha}, S^{\ a}_{\alpha}\} = \left\{\overline{S}_{\dot\alpha a}, \overline{S}_{\dot\alpha b}\right\} = 0$$

$$\left[K_\mu, S^{\ a}_{\alpha}\right] = \left[K_\mu, \overline{S}_{\dot\alpha a}\right] = 0$$

$$\boxed{\begin{array}{l} \left\{Q_{\alpha a}, S^{\ b}_{\beta}\right\} = -i\epsilon_{\alpha\beta}\sigma^{IJ\,b}_{\ \ \ a} R_{IJ} + \gamma^{\mu\nu}_{\alpha\beta}\delta^{b}_{a} L_{\mu\nu} - \frac{1}{2}\epsilon_{\alpha\beta}\delta^{b}_{a} D \\ \left\{\overline{Q}^{a}_{\dot\alpha}, \overline{S}_{\dot\beta b}\right\} = i\epsilon_{\dot\alpha\dot\beta}\sigma^{IJ\,a}_{\ \ \ b} R_{IJ} + \gamma^{\mu\nu}_{\dot\alpha\dot\beta}\delta^{a}_{b} L_{\mu\nu} - \frac{1}{2}\epsilon_{\dot\alpha\dot\beta}\delta^{a}_{b} D \end{array}}$$

- Conformal symmetry → No mass spectrum
- Conformal dimension spectrum for a local operator $[D,\mathcal{O}(0)] = -i\,\Delta\,\mathcal{O}(0)$
- Lowered by K : $\mathcal{O}'(0) \equiv \left[K_\mu,\mathcal{O}(0)\right] \to \left[D,\mathcal{O}'(0)\right] = -i\,(\Delta - 1)\,\mathcal{O}'(0)$

$$\left[D,K_\mu\right] = +i\,K_\mu$$

- Primary operator : $\left[K_\mu,\tilde{\mathcal{O}}(0)\right] = 0$
- Descendent operators : (ex) $\left[P_\mu,\tilde{\mathcal{O}}\right] = -i\partial_\mu\tilde{\mathcal{O}}$ $\quad\left[D,P_\mu\right] = -i\,P_\mu$

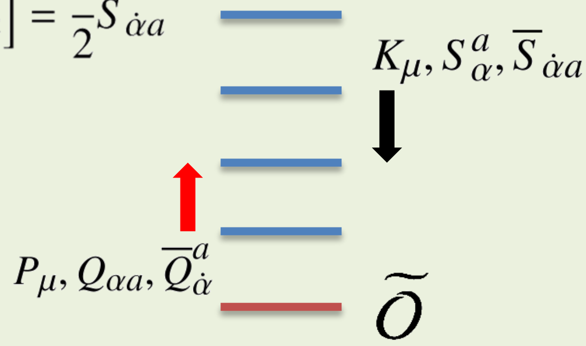
$$\left[D,\partial_\mu\tilde{\mathcal{O}}\right] = -i\,(\Delta + 1)\,\partial_\mu\tilde{\mathcal{O}}$$

- Superconformal raising and lowering ops.

$$[D,Q_{\alpha a}] = -\frac{i}{2}Q_{\alpha a},\quad \left[D,\overline{Q}^a_{\dot\alpha}\right] = -\frac{i}{2}\overline{Q}^a_{\dot\alpha},\quad [D,S^a_\alpha] = \frac{i}{2}S^a_\alpha,\quad \left[D,\overline{S}_{\dot\alpha a}\right] = \frac{i}{2}\overline{S}_{\dot\alpha a}$$

- Superconformal primary :

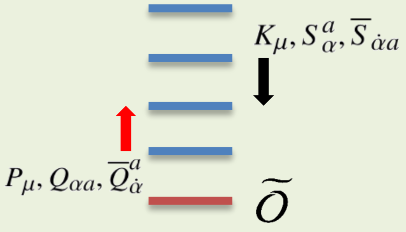
$$\left[S^a_\alpha,\tilde{\mathcal{O}}(0)\right] = \left[\overline{S}_{\dot\alpha a},\tilde{\mathcal{O}}(0)\right] = 0$$



- Cartan subalgebra $$[D,R]=\left[L_{\mu\nu},R\right]=\left[D,L_{\mu\nu}\right]=0$$
- Irreducible rep. are given by eigenvalues of these operators
$$\big(\overbrace{\Delta}^{D},\overbrace{S_1,S_2}^{L_{\mu\nu}}\,|\,\overbrace{J_1,J_2,J_3}^{R}\big)$$
- Scalar fields
$$Z\equiv\Phi_1+i\Phi_2,\quad Y\equiv\Phi_3+i\Phi_4,\quad X\equiv\Phi_5+i\Phi_6$$
$$\bar{Z}\equiv\Phi_1-i\Phi_2,\quad \bar{Y}\equiv\Phi_3-i\Phi_4,\quad \bar{X}\equiv\Phi_5-i\Phi_6$$
$$(1,0,0|\pm1,0,0),\ (1,0,0|0,\pm1,0),\ (1,0,0|0,0,\pm1)$$
- Gauginos and gauge fields
$$\overset{\chi^A_\alpha}{\left(\frac{3}{2},\pm\frac{1}{2},0|\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2}\right)},\ \overset{F_+}{(2,m,0|0,0,0)},\ \overset{\mathcal{D}}{\left(1,\pm\frac{1}{2},\pm\frac{1}{2}|0,0,0\right)}$$
- General gauge invariant composite operators
$$\tilde{\mathcal{O}}(x)=\mathrm{Tr}\left[\mathcal{O}_1(x)\mathcal{O}_2(x)\ldots\mathcal{O}_L(x)\right]$$
- ½ -BPS operator $\mathrm{Tr}\left[Z^L\right]\quad\rightarrow\quad(L,0,0|L,0,0)$

Chiral primary or BPS operator

- Impose further condition $\left[Q_{a\alpha}, \widetilde{O}(0)\right] = 0, \qquad \text{for some } \alpha, a$
  - Jacobi identity $\left[\left\{Q_{a\alpha}, S^b_\beta\right\}, \widetilde{O}(0)\right] = \left[-i\varepsilon_{\alpha\beta}\left(\sigma^{IJ}\right)^b_a R_{IJ} - \varepsilon_{\alpha\beta}\delta^b_a D + \sigma^{\mu\nu}_{\alpha\beta}\delta^b_a L_{\mu\nu}, \widetilde{O}(0)\right] = 0$
  - For the Lorentz scalar operator : $\left[L_{\mu\nu}, \widetilde{O}(0)\right] = 0$



$$\left(\sigma^{IJ}\right)^b_a \left[R_{IJ}, \widetilde{O}(0)\right] = \Delta\ \delta^b_a\ \widetilde{O}(0) \qquad \sigma^{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

  - Satisfied if R-charge = conformal dimension $\Delta = J_1$

$$\mathrm{Tr}\left[Z^L\right] \quad \rightarrow \quad (L, 0, 0 | L, 0, 0)$$

  - This commutes with half SUSY charges and conformal dimension is protected and gets no quantum corrections

Anomalous Dimension

- Conformal dimensions of composite operators :

$$\langle \mathcal{O}_n(x)\mathcal{O}_m(0)\rangle = \frac{\delta_{mn}}{|x|^{2\Delta_n}}$$

- Anomalous dimension is defined by $\Delta = \Delta_0 + \gamma$

- can be calculated by
  - Direct perturbation theory
  - Renormalization group under dilatation

- Operator mixing by RG dilatation

Perturbative computation

- (ex) Konishi operator $O(x) = \mathrm{Tr}\left(\sum_a \Phi_a(x)^2\right)$
- Tree-level

$$\left\langle :\Phi_a(x)^A_B\Phi_a(x)^B_A :: \Phi_b(y)^C_D\Phi_b(y)^D_C :\right\rangle_0 = \left(\frac{g^2_{\mathrm{YM}}}{8\pi^2}\right)^2 \frac{N_c^2\cdot 6\cdot 2}{|x-y|^4}$$

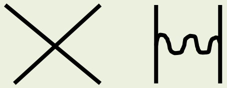
$$\langle O(x)O(y)\rangle = \frac{1}{|x-y|^{2\Delta}}$$

$$\frac{g^2_{\mathrm{YM}}}{8\pi^2}\frac{\delta^A_C\delta^D_B\delta_{ab}}{|x-y|^2}$$

$$\int \frac{d^4z}{|z-x|^4|z-y|^4} \approx \frac{2i}{|x-y|^4}\int_{\Lambda^{-1}}^{|x-y|}\frac{d\xi d\Omega_3}{\xi} = \frac{2\pi^2 i}{|x-y|^4}\ln(\Lambda^2|x-y|^2)$$

- One-loop

$$\left\langle :\Phi_a(x)^A_B\Phi_a(x)^B_A :: \Phi_b(y)^C_D\Phi_b(y)^D_C : \left(\frac{g^2_{\mathrm{YM}}}{4}\int d^4z\,\mathrm{Tr}(\Phi_c\Phi_c\Phi_d\Phi_d)(z)\right)\right\rangle_0 + \dots$$



$$\langle O_R(x)O_R(y)\rangle = \left(\frac{\lambda}{8\pi^2}\right)^2\frac{12}{|x-y|^4}\left[1-\frac{3\lambda}{4\pi^2}\ln(|x-y|^2)\right] \sim \frac{1}{|x-y|^{2(2+\underset{\gamma}{\boxed{3\lambda/4\pi^2}})}}$$

RG method

- Under dilatation $x \to \alpha x$

$$\mathcal{O}(x) = \alpha^{-\Delta}\mathcal{O}(\alpha x)$$

- One-loop corrections

- Operator mixing (ex) su(2) sector

$$\left\{\mathrm{Tr}\left[ZZZZXX\right], \mathrm{Tr}\left[ZZZXZX\right], \mathrm{Tr}\left[ZZXZZX\right], \mathrm{Tr}\left[ZXZZZX\right]\right\}$$

$$\mathcal{O}_a = \mathcal{Z}_a^b(\Lambda)\mathcal{O}_b$$

- Dilatation matrix

$$\Gamma = \frac{d\mathcal{Z}}{d\ln\Lambda} \cdot \mathcal{Z}^{-1}$$

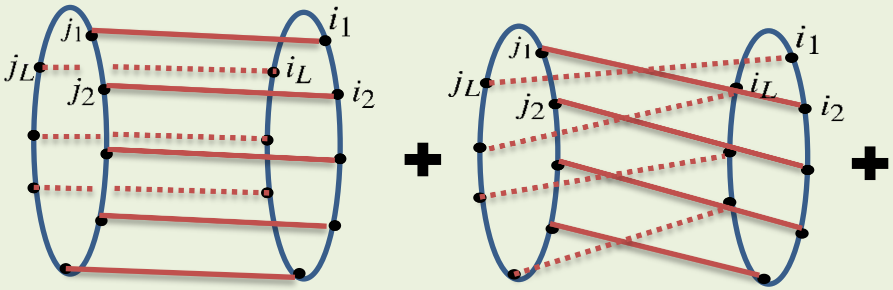
SO(6) sector

- Scalar fields $\{Z,\ Y,\ X,\ \overline{Z},\ \overline{Y},\ \overline{X}\}$
- Composite operators

$$\left\{\mathrm{Tr}\left[XYZ\overline{X}YZX\overline{Z}\cdots\right],\ldots\right\} = \mathrm{Tr}\left[\Phi_{i_1}\cdots\Phi_{i_L}\right] \equiv \mathcal{O}_{i_1\cdots i_L}(x)$$

- Two-point function $\left\langle \overline{\mathcal{O}}^{j_1\cdots j_L}(x)\,\mathcal{O}_{i_1\cdots i_L}(y)\right\rangle$

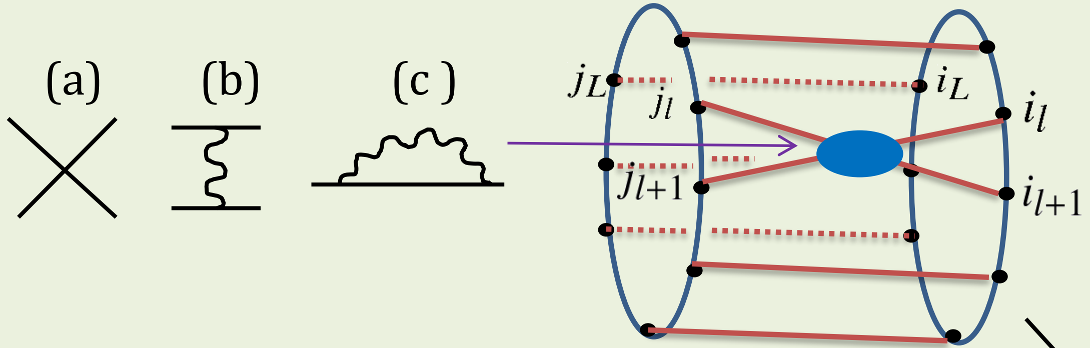
- Tree level

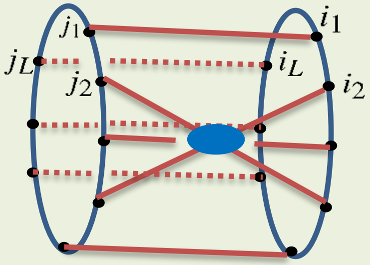


$$\left(\frac{\lambda}{8\pi^2}\right)^L \frac{1}{|x-y|^{2L}}\left[\delta^{j_1}_{i_1}\cdots\delta^{j_L}_{i_L} + \delta^{j_1}_{i_2}\cdots\delta^{j_L}_{i_1} + \ldots\right]$$

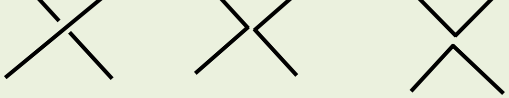
cyclic permutations

- One-loop level : nearest neighbor only





- Wave-function renormalization



$$Z^{(a)} = 1 - \frac{\lambda}{16\pi^2} \ln \Lambda \cdot \left(2\delta_{i_l}^{j_{l+1}} \delta_{i_{l+1}}^{j_l} - \delta_{i_l}^{j_l} \delta_{i_{l+1}}^{j_{l+1}} - \delta_{i_l, i_{l+1}} \delta^{j_l j_{l+1}} \right)$$

$$Z^{(b)} = 1 - \frac{\lambda}{16\pi^2} \ln \Lambda \cdot \delta_{i_l}^{j_l} \delta_{i_{l+1}}^{j_{l+1}}$$

$$Z^{(c)} = 1 + \frac{\lambda}{8\pi^2} \ln \Lambda \cdot \delta_{i_l}^{j_l} \delta_{i_{l+1}}^{j_{l+1}}$$

$$Z = 1 + \frac{\lambda}{16\pi^2} \ln \Lambda \cdot \left(\delta_{i_l, i_{l+1}} \delta^{j_l j_{l+1}} - 2\delta_{i_l}^{j_{l+1}} \delta_{i_{l+1}}^{j_l} + 2\delta_{i_l}^{j_l} \delta_{i_{l+1}}^{j_{l+1}} \right)$$

- Dilatation matrix

Minahan, Zarembo (2003)

$$\Gamma = \frac{\lambda}{8\pi^2} \sum_{l=1}^{L} \left(1 - \mathbf{P}_{l,l+1} + \frac{1}{2} \mathbf{K}_{l,l+1} \right)$$

Mapping to integrable spin chain

- Finding the eigenvalues of the dilatation matrix is very difficult problem but fortunately …
- Mapping the matrix to a Hamiltonian of integrable spin chain has been discovered [(ex) so(6), su(2) spin chains]
- (ex) su(2) sector $\left\{ \mathrm{Tr}\left[Z^L\right], \mathrm{Tr}\left[Z^{L-1}X\right], \mathrm{Tr}\left[Z^{L-n-1}XZ^{n-1}X\right], \ldots, \mathrm{Tr}\left[X^L\right] \right\}$
- One-loop dilatation → Heisenberg spin chain model
  - Map: $|\Uparrow\rangle \equiv |Z\rangle, \quad |\Downarrow\rangle \equiv |X\rangle$
  - Vacuum state: BPS $|0\rangle \equiv \mathrm{Tr}\left[Z^L\right]$
  - Excited states:

$$\Gamma = \frac{\lambda}{8\pi^2} \sum_{l=1}^{L} [1 - \vec{\sigma}_l \cdot \vec{\sigma}_{l+1}]$$

$$|\Uparrow\Downarrow\Uparrow\Downarrow\Uparrow\Uparrow\ldots\rangle + \ldots \equiv \mathrm{Tr}\,[ZXZXZZ + \ldots] + \ldots$$

Heisenberg model

- 1D spin chain, XXX model

$$H = \sum_{l=1}^{L} \left(1 - \vec{\sigma}_l \cdot \vec{\sigma}_{l+1}\right)$$

$$\sigma_j^a = \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \sigma^a \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} : \quad 2^L \times 2^L \ \text{Matrix}$$

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Can be exactly solvable

- Reference : Lecture by Rafael Nepomechie

Bethe ansatz equation

- Eigenvectors

$$|p_1, p_2, \cdots\rangle = \sum_{n_1, n_2, \cdots = 1}^{L} A(p_1, p_2, \cdots) e^{i(n_1 p_1 + n_2 p_2 + \cdots)} | \cdots \Uparrow \underset{n_1}{\Downarrow} \Uparrow \cdots \Uparrow \underset{n_2}{\Downarrow} \Uparrow \cdots \rangle + \cdots$$

- BAE

$$e^{ip_j L} = \prod_{\substack{k=1 \\ k \neq j}}^{M} \frac{\cot\frac{p_j}{2} - \cot\frac{p_k}{2} + 2i}{\cot\frac{p_j}{2} - \cot\frac{p_k}{2} - 2i}, \qquad j = 1, \ldots, M$$

$$\left(\frac{u_j + i/2}{u_j - i/2}\right)^L = \prod_{\substack{k=1 \\ k \neq j}}^{M} \frac{u_j - u_k + i}{u_j - u_k - i}, \qquad \frac{u + i/2}{u - i/2} \equiv e^{ip}$$

- Anomalous dimensions are given by the eigenvalues

$$\gamma = \frac{\lambda}{2\pi^2} \sum_{j=1}^{M} \sin^2 \frac{p_j}{2} = \frac{\lambda}{2\pi^2} \sum_{j=1}^{M} \frac{1}{u_j^2 + \frac{1}{4}}$$

- Actual solution of BAE for generic M, L is non-trivial

- Cyclicity of the trace : $n_j \to n_j + L$

$$\sum_{j=1}^{M} p_j = 0$$

- (Ex) Two “magnon” state :

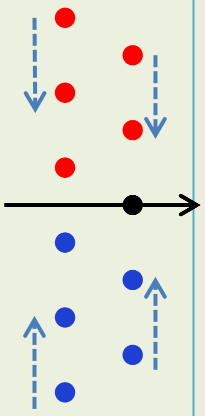
$$|\Uparrow\Uparrow\Downarrow\Downarrow\Uparrow\cdots\Uparrow\Downarrow\Downarrow\rangle + \ldots \equiv \mathrm{Tr}\left[X^2 Z^{N-2} + \ldots\right]$$

$$\left(\frac{u_1 + \frac{i}{2}}{u_1 - \frac{i}{2}}\right)^L = \frac{u_1 - u_2 + i}{u_1 - u_2 - i} = \frac{u_1 + \frac{i}{2}}{u_1 - \frac{i}{2}} \quad \text{with} \quad u_1 = -u_2$$

$$\gamma = \frac{\lambda}{\pi^2}\sin^2\frac{n\pi}{L-1} \quad \xrightarrow{L\gg 1} \quad \frac{n^2\lambda}{L^2}$$

Bethe Strings

- Bethe roots so far were real but complex roots can exist
- L→∞ limit:
  - Introduce a complex pair of root with imaginary parts $u_j = u^R \pm i\alpha$
  - For positive imaginary root: LHS of BAE $\left(\frac{u_j + i/2 + i\alpha}{u_j - i/2 + i\alpha}\right)^L \to \infty$
  - RHS of BAE : there should be another Bethe root which makes a denominator vanish : $u_k = u^R + i(\alpha - 1)$
  - Repeat the process until the imaginary part is still positive
  - For negative imaginary root: LHS of BAE $\left(\frac{u_j + i/2 - i\alpha}{u_j - i/2 - i\alpha}\right)^L \to 0$
  - RHS should vanish by adding $u_l = u^R + i(-\alpha + 1)$
  - For finite # of roots, α should be an integer or a half-integer

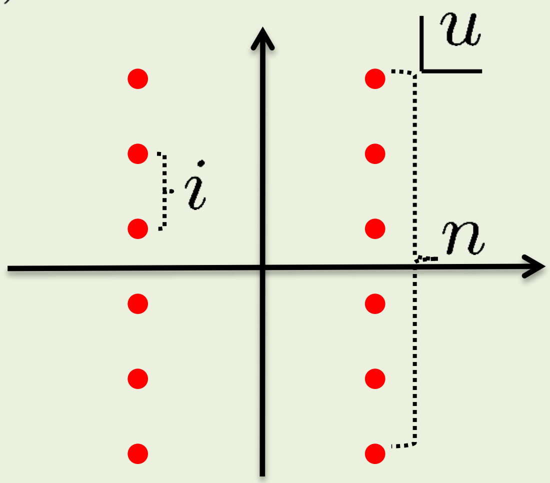


- Bethe string

$$u_j^{(n)} = u^R + \frac{n+1-2j}{2} i, \quad j = 1, \ldots, n$$

- For finite L : strings are deformed
- Low lying states are given by “long strings” rather than real roots

$$E^{(n)} = \sum_{j=1}^{n} \frac{1}{\left(u_j^{(n)}\right)^2 + \frac{1}{4}} = \frac{n}{(u^R)^2 + \frac{n^2}{4}} \leq \frac{n}{(u^R)^2 + \frac{1}{4}} = nE^{(1)}$$

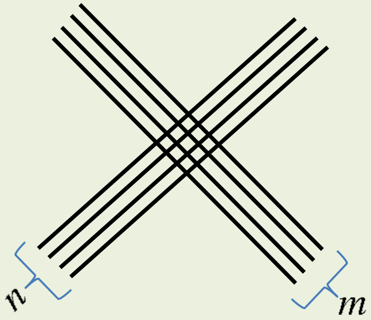


- BAE for the strings can be obtained by multiplying elementary BAE for each component of a string
  - Elementary BAE

$$e_1(u_j)^L = \prod_{k \neq j}^{M} e_2(u_j - u_k) \qquad e_n(u) \equiv \frac{u + in/2}{u - in/2}$$

  - BAE for strings

$$\prod_{j=1}^{n} \frac{u^R + i(n+1-2j)/2 + i/2}{u^R + i(n+1-2j)/2 - i/2} = \frac{u^R + in/2}{u^R - in/2} = e_n(u^R)$$



$$e_{n_J}(u_J^R)^L = \prod_{K=1}^{M} E_{n_J,n_K}(u_J^R - u_K^R) \qquad E_{n,m} = e_{|n-m|} e^2_{|n-m|+2} \cdots e^2_{n+m-2} e_{n+m}$$

SO(6) BAE

- so(6) is also integrable
- 3 sets of Bethe roots

$$e_1(u_j^1)^L = \prod_{k\neq j}^{n_1} e_2(u_j^1-u_k^1)\prod_{k}^{n_2} e_{-1}(u_j^1-u_k^2)\prod_{k}^{n_3} e_{-1}(u_j^1-u_k^3)$$

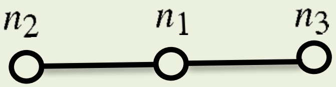
$$1 = \prod_{k\neq j}^{n_2} e_2(u_j^2-u_k^2)\prod_{k}^{n_1} e_{-1}(u_j^2-u_k^1)$$

$$1 = \prod_{k\neq j}^{n_3} e_2(u_j^3-u_k^3)\prod_{k}^{n_1} e_{-1}(u_j^3-u_k^1)$$

$$e_n(u) \equiv \frac{u+in/2}{u-in/2}$$

- su(4) Dynkin

$$\vec\mu=(1,0,0),\ \vec\alpha_1=(1,-1,0),\ \vec\alpha_2=(0,1,-1),\ \vec\alpha_3=(0,1,1)$$



$$Z=(1,0,0)=\vec\mu$$
$$X=(0,1,0)=\vec\mu-\vec\alpha_1$$
$$Y=(0,0,1)=\vec\mu-\vec\alpha_1-\vec\alpha_2$$
$$\bar Z=(-1,0,0)=\vec\mu-2\vec\alpha_1-\vec\alpha_2-\vec\alpha_3$$
$$\bar X=(0,-1,0)=\vec\mu-\vec\alpha_1-\vec\alpha_2-\vec\alpha_3$$
$$\bar Y=(0,0,-1)=\vec\mu-\vec\alpha_1-\vec\alpha_3$$

$$\bar Z=(X\bar X)\text{ or }(Y\bar Y)$$

$$e_{\vec\alpha_a\cdot\vec\mu}(u_j^a)^L=\prod_{b}\prod_{\substack{k=1\\(b,k)\neq(a,j)}}^{n_b} e_{\vec\alpha_a\cdot\vec\alpha_b}(u_j^a-u_k^b)$$

$$\vec w = L\vec\mu - n_1\vec\alpha_1 - n_2\vec\alpha_2 - n_3\vec\alpha_3$$

$$(J_1,J_2,J_3)=(L-n_1,n_1-n_2-n_3,n_2-n_3)$$

$$n_3=0\ \rightarrow\ \mathrm{Tr}\left[\overbrace{Z\cdots Z}^{L-n_1}\overbrace{X\cdots X}^{n_1-n_2}\overbrace{Y\cdots Y}^{n_2}\right]+\text{perm.}$$

- (ex) $n_1=2$, $n_2=1$

$$\mathrm{Tr}\left[\overbrace{Z\cdots Z}^{L-2}XY\right]+\text{perm.}\qquad u_1^1,\ u_2^1,\ u^2$$

- Total momentum=0 → $u_1^1=-u_2^1$

- BAE: $e_1(u_1^1)^L=\cancel{e_2(2u_1^1)}\cancel{e_{-1}(u_1^1-u^2)},\ 1=e_{-1}(u^2-u_1^1)e_{-1}(u^2+u_1^1)$

$$\downarrow$$

$$u^2=0\quad\text{or}\quad\infty$$

  - $u^2=0$: XY-YX

$$e_1(u_1^1)^L=1\quad\rightarrow\quad p_1=\frac{2\pi n}{L},\quad\gamma=\frac{\lambda}{2\pi^2}\sin^2\frac{\pi n}{L}$$

  - $u^2=\infty$: XY+YX

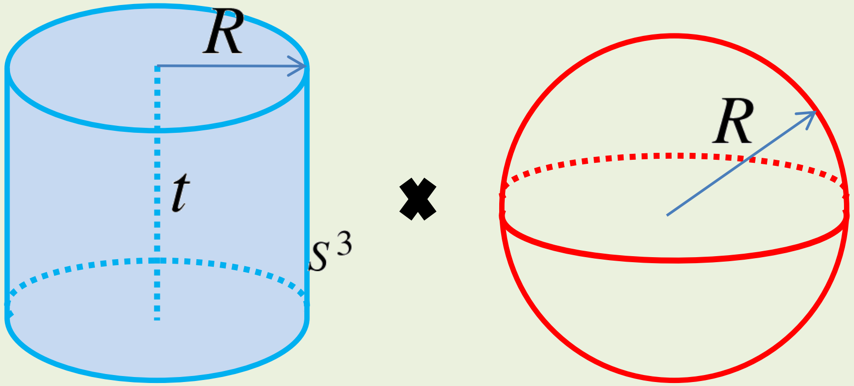
$$e_1(u_1^1)^{L-1}=1\quad\rightarrow\quad p_1=\frac{2\pi n}{L-1},\quad\gamma=\frac{\lambda}{2\pi^2}\sin^2\frac{\pi n}{L-1}$$

Classical integrability in string theory

Superstring on AdS background

- Type IIB superstrings on $AdS_5 \times S^5$ is described by

$$S = \frac{R^2}{\alpha'} \int d\tau d\sigma \left[G_{mn}^{(S^5)} \partial_a X^m \partial^a X^n + G_{mn}^{(AdS)} \partial_a Y^m \partial^a Y^n + \text{fermions} \right]$$



Metsaev, Tseytlin (1998)

Bena, Polchinski, Roiban (2003)

- Virasoro constraints $\dot{X}^m X'_m + \dot{Y}^n Y'_n = 0, \quad \dot{X}^m \dot{X}_m + \dot{Y}^n \dot{Y}_n + X'^m X'_m + Y'^n Y'_n = 0$
- Classically integrable nonlinear sigma model on coset

$$\mathrm{AdS}_5 \times S^5 \approx SO(4,2)/SO(4,1) \times SO(6)/SO(5) \quad \rightarrow \quad PSU(2,2|4)/[SO(4,1) \times SO(5)]$$

- $AdS_5 \times S^5$ space $\quad X_1^2 + \cdots + X_6^2 = 1, \qquad Y_0^2 - Y_1^2 - \cdots - Y_4^2 + Y_5^2 = 1$
- Global coordinates

$$Y_1 + iY_2 = \sinh\rho \cos\psi e^{i\phi_1}, \qquad Y_3 + iY_4 = \sinh\rho \sin\psi e^{i\phi_2},$$
$$Y_5 + iY_0 = \cosh\rho e^{it}, \qquad X_5 + iX_6 = \cos\gamma e^{i\varphi_3},$$
$$X_1 + iX_2 = \sin\gamma \cos\theta e^{i\varphi_1}, \qquad X_3 + iX_4 = \sin\gamma \sin\theta e^{i\varphi_2}$$
$$\begin{aligned}(ds^2)_{AdS_5} &= R^2\left[d\rho^2 - \cosh^2\rho dt^2 + \sinh^2\rho(d\psi^2 + \cos^2\psi d\phi_1^2 + \sin^2\psi d\phi_2^2)\right] \\ (ds^2)_{S^5} &= R^2\left[d\gamma^2 + \cos^2\gamma d\varphi_3^2 + \sin^2\gamma(d\theta^2 + \cos^2\theta d\varphi_1^2 + \sin^2\theta d\varphi_2^2)\right]\end{aligned}$$

- Eqs. of motion

$$\begin{aligned}\partial^a\partial_a Y_n - \tilde{\Lambda} Y_n &= 0, \qquad \tilde{\Lambda} = \partial^a Y_n \partial_a Y^n, \qquad Y_n Y^n = -1 \\ \partial^a\partial_a X_m - \Lambda X_m &= 0, \qquad \Lambda = \partial^a X_m \partial_a X^m, \qquad X_m X^m = 1\end{aligned}$$

- 6 isometry coordinates $\quad t,\ \phi_1,\ \phi_2,\ \varphi_1,\ \varphi_2,\ \varphi_3$
- Conserved charges

$$S_{pq} = \sqrt{\lambda}\int_0^{2\pi}\frac{d\sigma}{2\pi}(Y_p\dot{Y}_q - Y_q\dot{Y}_q), \quad J_{mn} = \sqrt{\lambda}\int_0^{2\pi}\frac{d\sigma}{2\pi}(X_m\dot{X}_n - X_n\dot{X}_m)$$
$$(S_{50}, S_{12}, S_{34}|J_{12}, J_{34}, J_{56}) \quad \leftrightarrow \quad (\Delta, S_1, S_2|J_1, J_2, J_3)$$

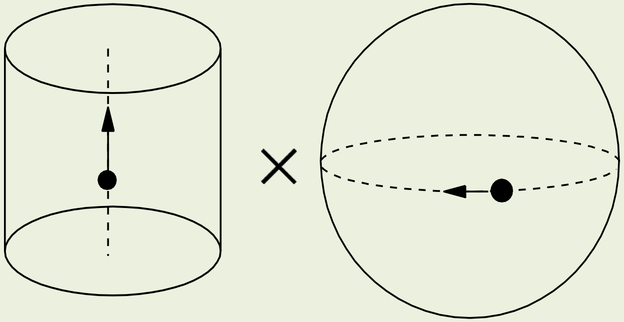
BPS string

- A point-like [no σ dependence] string which rotates on a great circle of S^5 with angular momentum $J \to \infty$

$$Y_5+iY_0 = e^{i\kappa\tau},\ X_1+iX_2 = e^{i\kappa\tau},\ \kappa = \sqrt{\Lambda},\ Y_{1,2,3,4} = X_{3,4,5,6} = 0$$

- Energy = angular momentum

$$E = J_1 = \sqrt{\lambda}\kappa$$



BMN string

Berenstein, Maldacena, Nastase (2002)

- A point-like string in the planar-limit $R \to \infty$ with $\rho,\ \gamma \to 0$

$$\begin{aligned} ds^2 &= R^2\left[d\rho^2 - (1+\rho^2)dt^2 + d\gamma^2 + (1-\gamma^2)d\varphi_3^2 + \rho^2(d\Omega_3)^2 + \gamma^2(d\Omega_3')^2\right] + O(R^{-2}) \\ &= R^2(d\varphi_3^2 - dt^2) + dr^2 - r^2dt^2 + dy^2 - y^2d\varphi_3^2 + r^2(d\Omega_3)^2 + y^2(d\Omega_3')^2 \quad (r \equiv R\rho,\ y \equiv R\gamma) \\ &= -2dx^+dx^- - \mu^2(r^2+y^2)(dx^+)^2 + dr^2 + dy^2 + r^2(d\Omega_3)^2 + y^2(d\Omega_3')^2 \qquad t, \varphi_3 = \mu x^+ \pm \frac{x^-}{\mu R^2} \end{aligned}$$

- Moving on the null geodesic $t = \varphi_3$
- Orthogonal directions have only quadratic fluctuations

$$S = \sqrt{\lambda}\int d^2\xi\left[\frac{1}{2}(\partial_a x^i)^2 - \frac{(\mu\alpha' p^+)^2}{2}(x^i)^2\right] \quad \text{with} \quad p^+ = \frac{J}{\mu R^2}$$

- Energy $E - J = \sum_{n=-\infty}^{\infty} N_n \sqrt{1 + \frac{\lambda}{J^2}n^2}$ exact in all orders of λ

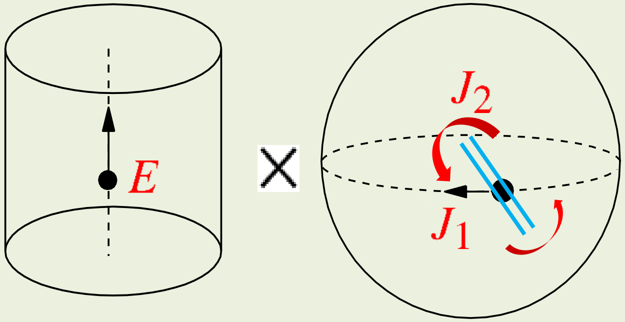
- Relation to the SYM

$$|0\rangle \quad \leftrightarrow \quad \mathrm{Tr}\left[Z^J\right], \qquad E = J$$

$$a_{-n}^i a_n^i|0\rangle \quad \leftrightarrow \quad \mathrm{Tr}\left[X^2Z^J + \ldots\right], \qquad E = J + 2\sqrt{1 + \lambda n^2/J^2}$$

Folded string

Gubser,Klebanov,Polyakov ; Frolov, Tseytlin (2002)



- SU(2) sector of SYM maps to
- Consider a folded string which is spinning in $R_t \times S^3 \rightarrow \rho = 0,\ \gamma = \frac{\pi}{2},\ \varphi_3 = 0$

$$t = \kappa\tau,\ \varphi_1 = \omega_1\tau,\ \varphi_2 = \omega_2\tau,\ -\theta_0 \le \theta(\sigma) \le \theta_0$$

- Conserved charges :

$$E = \sqrt{\lambda}\kappa,\ J_1 = \sqrt{\lambda}\omega_1 \int_0^{2\pi} \frac{d\sigma}{2\pi} \cos^2\theta(\sigma),\ J_2 = \sqrt{\lambda}\omega_2 \int_0^{2\pi} \frac{d\sigma}{2\pi} \sin^2\theta(\sigma)$$

- Effective 1d action:

$$S = \frac{\sqrt{\lambda}}{4\pi} \int_0^{2\pi} d\sigma \left[\kappa^2 + \theta'(\sigma)^2 - \omega_1^2 \cos^2\theta(\sigma) - \omega_2^2 \sin^2\theta(\sigma)\right]$$

$$E = J + \frac{2\lambda}{J\pi^2}\mathbf{K}(q_0)\left[\mathbf{E}(q_0) - (1-q_0)\mathbf{K}(q_0)\right], \quad \frac{J_2}{J} = 1 - \frac{\mathbf{E}(q_0)}{\mathbf{K}(q_0)}$$

Neumann-Rosochatius reduction

Arutyunov, Russo, Tseytlin (2002)

- A string in $R_t \times S^3 \;\rightarrow\; \rho = 0,\ \gamma = \frac{\pi}{2},\ \varphi_3 = 0$

$$t = \kappa\tau,\ \cos\theta(\sigma,\tau) = r_1(\xi),\ \sin\theta(\sigma,\tau) = r_2(\xi),\ \varphi_j(\sigma,\tau) = \omega_j\tau + f_j(\xi),\ \xi = \alpha\sigma + \beta\tau$$

- Effective 1d Lagrangian (Neumann-Rosochatius):

$$L_{NR} = (\alpha^2 - \beta^2)\sum_{j=1}^{2}\left[r_j'^2 - \frac{1}{(\alpha^2-\beta^2)^2}\left(\frac{C_j^2}{r_j^2} + \alpha^2\omega_j^2 r_j^2\right)\right] + \Lambda\left(\sum_{j=1}^{2} r_j^2 - 1\right)$$

- Conserved charges and Virasoro constraints

$$E = \frac{\lambda}{2\pi}\frac{\kappa}{\alpha}\int d\xi, \quad J_j = \frac{\lambda}{2\pi}\frac{1}{\alpha^2-\beta^2}\int d\xi\left(\frac{\beta}{\alpha}C_j + \alpha\omega_j r_j^2\right), \qquad \sum_{j=1}^{2} C_j\omega_j + \beta\kappa^2 = 0$$

- Exact solution $E = \frac{\sqrt{\lambda}}{2\pi}\mathcal{E}, \quad J = \frac{\sqrt{\lambda}}{2\pi}\mathcal{J}, \quad v = -\frac{\beta}{\alpha}$

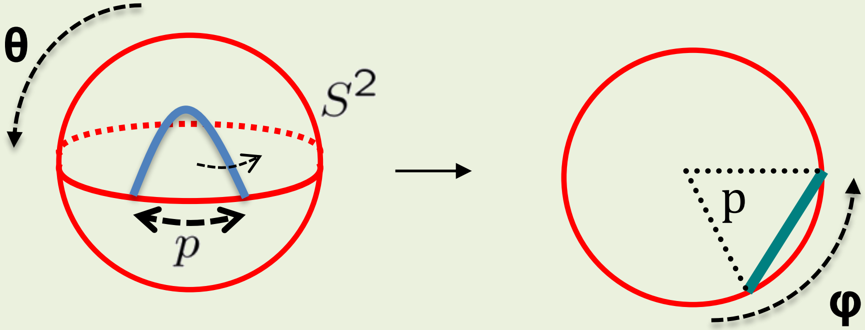
$$\begin{aligned}
\mathcal{E} &= 2\sqrt{(1-v^2)(1-\epsilon)}\mathbf{K}(1-\epsilon), \quad \mathcal{J} = 2\sqrt{\frac{1-v^2}{1-v^2\epsilon}}\left[\mathbf{K}(1-\epsilon)-\mathbf{E}(1-\epsilon)\right], \\
\mathcal{E}-\mathcal{J} &= 2\sqrt{\frac{1-v^2}{1-v^2\epsilon}}\left[\mathbf{E}(1-\epsilon)-\left(1-\sqrt{(1-v^2\epsilon)(1-\epsilon)}\right)\mathbf{K}(1-\epsilon)\right], \\
p &= 2v\sqrt{\frac{1-v^2\epsilon}{1-v^2}}\left[\frac{1}{v^2}\Pi\left(1-\frac{1}{v^2}|1-\epsilon\right)-\mathbf{K}(1-\epsilon)\right]
\end{aligned}$$

- Infinite J limit

$$\underbrace{\mathcal{E}-\mathcal{J} = 2\sin(p/2)}_{\text{Giant magnon}}\left[1\overset{\text{Finite-size correction}}{\boxed{-4\sin^2(p/2)\exp\left(-\frac{\mathcal{J}}{\sin(p/2)}-2\right)}}\right]$$

Giant magnon

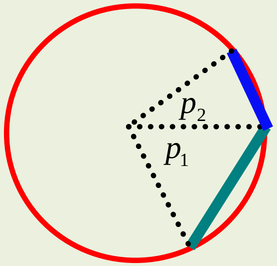
- Classical string configuration in R x S^2 **Hofman, Maldacena (2006)**



$$\cos\theta = \frac{\sin\frac{p}{2}}{\cosh\xi}, \qquad \tan\varphi = \tan\frac{p}{2}\tanh\xi, \qquad \xi \equiv \frac{\sigma - \cos\frac{p}{2}\tau}{\sin\frac{p}{2}}$$

  - Energy of the string $E = \frac{\sqrt{\lambda}}{\pi}\sin\frac{p}{2}$
  - S^2 angle θ is related to the sine-Gordon field, GM is mapped to the SG "Soliton" $\Phi = 2\tan^{-1}(e^{\xi})$
  - Dual to magnons in the SYM spin chain $\cdots \Uparrow\Uparrow \Downarrow \Uparrow\Uparrow \cdots$

Two soliton configuration



- Two soliton scattering amplitude from time delay:

$$S(p_1,p_2)=e^{i\delta(p_1,p_2)},\qquad \delta(p_1,p_2)=4g\left(\cos\frac{p_1}{2}-\cos\frac{p_2}{2}\right)\log\frac{\sin^2\frac{p_1+p_2}{4}}{\sin^2\frac{p_1-p_2}{4}}$$

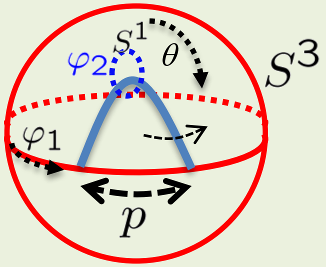
- Should be compared with strong coupling limit of exact S-matrix

Dyonic giant magnon

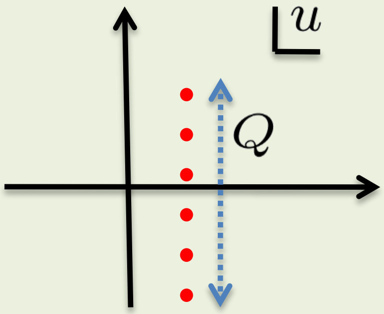
Chen, Dorey, Okamura (2007)

- GM in $R \times S^3 \to |Z_1|^2 + |Z_2|^2 = 1 \;\; (Z_1 = \sin\theta e^{i(\tau+\varphi_1)}, \quad Z_2 = \cos\theta e^{i(\omega\tau+\varphi_2)})$

$$\cos\theta = \frac{\sin\frac{p}{2}}{\cosh\tilde{\xi}}, \qquad \tan\varphi_1 = \tan\frac{p}{2}\tanh\tilde{\xi}, \qquad \tilde{\xi} \equiv \alpha\sigma + \beta\tau$$



  - Related to classical complex sine-Gordon model
  - Energy-charge relation: $E - J_1 = \sqrt{Q^2 + \frac{\lambda}{\pi^2}\sin^2\frac{p}{2}}, \quad J_2 = Q \quad Q \sim \sqrt{\lambda} >> 1$
  - Dual to magnon bound states “Bethe string”



All-Loop Bethe ansatz

- Conjecture: su(2) sector

Beisert, Dipple, Staudacher (2004)

$$e^{ip_jL} = \prod_{\substack{k=1\\k\neq j}}^{M}\left[\sigma^2(x_j,x_k)\frac{u_j-u_k+i}{u_j-u_k-i}\right]$$

$$x_j^\pm = e^{\pm i\frac{p_j}{2}}\left[\frac{1+\sqrt{1+16g^2\sin^2\frac{p_j}{2}}}{4g\sin\frac{p_j}{2}}\right], \quad x_j^\pm+\frac{1}{x_j^\pm} = u_j\pm\frac{i}{2g}, \quad u_j = \frac{1}{2}\cot\frac{p_j}{2}\sqrt{1+16g^2\sin^2\frac{p_j}{2}} \qquad g\equiv\frac{\sqrt{\lambda}}{4\pi}$$

$$\Delta = M+\gamma = \sum_{j=1}^{M}\sqrt{1+16g^2\sin^2\frac{p_j}{2}}$$

$$\xrightarrow{g\ll 1\text{ limit}} \frac{\lambda}{2\pi^2}\sin^2\frac{p_j}{2}$$

$$\xrightarrow{g\gg 1\text{ limit}} \frac{\lambda}{\pi}\sin\frac{p_j}{2}$$

$$\xrightarrow{\text{BMN limit}} \sqrt{1+\frac{\lambda}{J^2}n_j^2} \qquad \left(p_j=\frac{2\pi n_j}{J}\right)$$

- Matches well with perturbative computations up to 4 loops

BES dressing factor

Beisert-Hernandez-Lopez, Beisert-Eden-Staudacher

- Integral Representation: (derivation later)

Dorey, Hofman, Maldacena (2006)

$$\chi(x,y) = -\chi(y,x)$$

$$\sigma(x_1, x_2) \;=\; \exp\left\{i\left[\chi(x_1^+, x_2^-) + \chi(x_1^-, x_2^+) - \chi(x_1^+, x_2^+) - \chi(x_1^-, x_2^-)\right]\right\}$$

$$\chi(x,y) \;=\; -i\oint_{|z|=1}\frac{dz}{2\pi i}\oint_{|z'|=1}\frac{dz'}{2\pi i}\frac{1}{x-z}\frac{1}{y-z'}\frac{\ln\Gamma\left[1+ig\left(z_1+\frac{1}{z_1}-z_2-\frac{1}{z_2}\right)\right]}{\ln\Gamma\left[1-ig\left(z_1+\frac{1}{z_1}-z_2-\frac{1}{z_2}\right)\right]}$$

- Match with all the existing approximate results including classical string theory

Arutyunov, Frolov, Staudacher

Three-loop su(2) Konishi

- su(2) Konishi $\mathrm{Tr}\,[ZZXX]\,, \quad \mathrm{Tr}\,[ZXZX]$

- BAE : $p_1 = -p_2 = p, \quad \sigma \approx 1 + \mathcal{O}(g^6)$

$$e^{i4p} = \frac{2u+i}{2u-i}, \qquad u = \frac{1}{2}\cot\frac{p}{2}\sqrt{1+16g^2\sin^2\frac{p}{2}}$$

- Perturbative solutions $p = \frac{2\pi}{3} - \sqrt{3}g^2 + \frac{9\sqrt{3}}{2}g^4 + \ldots$

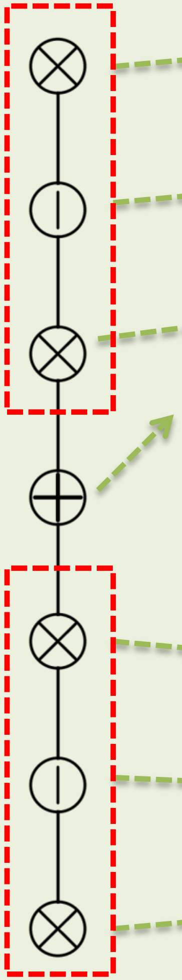
- One gets

$$\Delta = 4 + \overbrace{\underset{\overset{\uparrow}{\frac{3}{4\pi^2}\lambda}}{12g^2} - 48g^4 + 336g^6}^{\text{Match with perturbative SYM}} + \mathcal{O}(g^8)$$

Full sector Conjecture

Beisert-Staudacher

$$\mathcal{O}(x) = \mathrm{Tr}\Big[\ldots ZX \ldots ZY \ldots ZF_{\mu\nu} \ldots Z\chi^{\alpha} \ldots ZD_{\mu}Y \ldots\Big]$$



$$1 = \prod_{k=1}^{K_2} \frac{u_{1j} - u_{2k} + \frac{i}{2}}{u_{1j} - u_{2k} - \frac{i}{2}} \prod_{k=1}^{K_4} \frac{1 - 1/x_{1j}x_{4k}^{+}}{1 - 1/x_{1j}x_{4k}^{-}}$$

$$1 = \prod_{k=1}^{K_2} \frac{u_{2j} - u_{2k} - i}{u_{2j} - u_{2k} + i} \prod_{k=1}^{K_3} \frac{u_{2j} - u_{3k} + \frac{i}{2}}{u_{2j} - u_{3k} - \frac{i}{2}} \prod_{k=1}^{K_1} \frac{u_{2j} - u_{1k} + \frac{i}{2}}{u_{2j} - u_{1k} - \frac{i}{2}}$$

$$1 = \prod_{k=1}^{K_2} \frac{u_{3j} - u_{2k} + \frac{i}{2}}{u_{3j} - u_{2k} - \frac{i}{2}} \prod_{k=1}^{K_4} \frac{x_{3j} - x_{4k}^{+}}{x_{3j} - x_{4k}^{-}}$$

$$\left(\frac{x_{4j}^{+}}{x_{4j}^{-}}\right)^{L} = \prod_{k=1}^{K_4} \sigma^2(x_{4j}, x_{4k}) \frac{u_{4j} - u_{4k} + i}{u_{4j} - u_{4k} - i}$$

$$\times \prod_{k=1}^{K_1} \frac{1 - 1/x_{4j}^{-}x_{1k}}{1 - 1/x_{4j}^{+}x_{1k}} \prod_{k=1}^{K_3} \frac{x_{4j}^{-} - x_{3k}}{x_{4j}^{+} - x_{3k}} \prod_{k=1}^{K_5} \frac{x_{4j}^{-} - x_{5k}}{x_{4j}^{+} - x_{5k}} \prod_{k=1}^{K_7} \frac{1 - 1/x_{4j}^{-}x_{7k}}{1 - 1/x_{4j}^{+}x_{7k}}$$

$$1 = \prod_{k=1}^{K_6} \frac{u_{5j} - u_{6k} + \frac{i}{2}}{u_{5j} - u_{6k} - \frac{i}{2}} \prod_{k=1}^{K_4} \frac{x_{5j} - x_{4k}^{+}}{x_{5j} - x_{4k}^{-}}$$

$$1 = \prod_{k=1}^{K_6} \frac{u_{6j} - u_{6k} - i}{u_{6j} - u_{6k} + i} \prod_{k=1}^{K_5} \frac{u_{6j} - u_{5k} + \frac{i}{2}}{u_{6j} - u_{5k} - \frac{i}{2}} \prod_{k=1}^{K_7} \frac{u_{6j} - u_{7k} + \frac{i}{2}}{u_{6j} - u_{7k} - \frac{i}{2}}$$

$$1 = \prod_{k=1}^{K_6} \frac{u_{7j} - u_{6k} + \frac{i}{2}}{u_{7j} - u_{6k} - \frac{i}{2}} \prod_{k=1}^{K_4} \frac{1 - 1/x_{7j}x_{4k}^{+}}{1 - 1/x_{7j}x_{4k}^{-}}$$

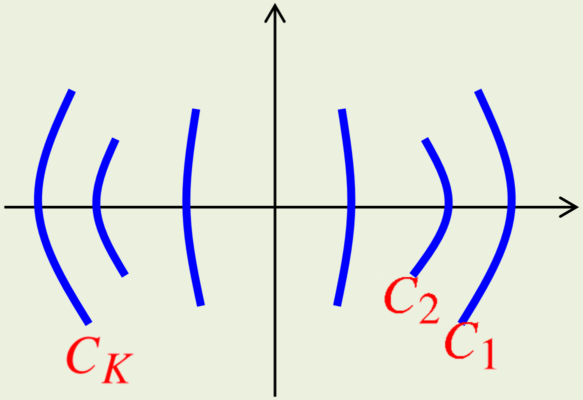
Algebraic curves

strong coupling limit of su(2) BAE

- (ex) Thermodynamic limit of su(2) BAE $L,\ M \to \infty, \quad u_j \sim L$

$$L \ln \frac{u_j + i/2}{u_j - i/2} = \sum_{k \neq j}^{M} \ln \frac{u_j - u_k + i}{u_j - u_k - i} - 2\pi i n_j \quad \xrightarrow[x_j \equiv \frac{u_j}{L}]{} \quad \frac{1}{x_j} = \frac{2}{L} \sum_{k \neq j}^{M} \frac{1}{x_j - x_k} - 2\pi n_j$$

- Bethe roots condensate and form cuts on the complex plane



$C \equiv C_1 \cup \ldots \cup C_K$

- Define density of roots

$$\rho(x) = \frac{1}{L} \sum_{j-1}^{M} \delta(x - x_j)$$

- Energy and momentum :

$$\gamma = \frac{\lambda}{8\pi^2 L^2}\sum_{j=1}^{M}\frac{1}{x_j^2} = \frac{\lambda}{8\pi^2 L}\int_C \frac{\rho(x)}{x^2}dx, \qquad P = \frac{1}{L}\sum_{j=1}^{M}\frac{1}{x_j} = \int_C \frac{\rho(x)}{x}dx = 2\pi m$$

- Continuum BAE : $\frac{1}{x} = 2\int_C dy \frac{\rho(y)}{x-y\pm i0} \pm 2\pi i\rho(x) - 2\pi n_x$

- Resolvent :

$$G(x) = \int_C dy \frac{\rho(y)}{x-y} = \frac{1}{L}\sum \frac{1}{x-x_k}$$

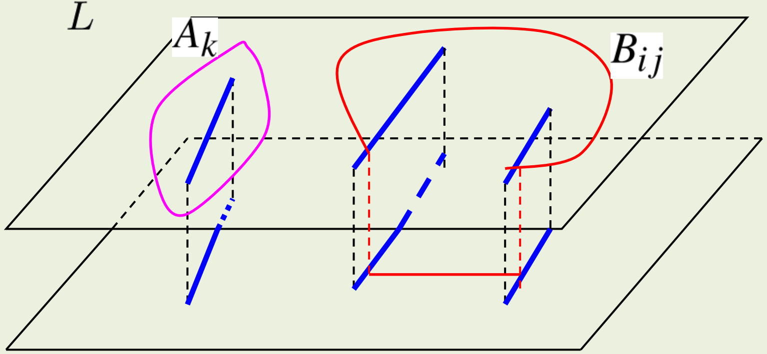
$$G(x+i0)+G(x-i0) = \frac{1}{x}+2\pi n_x, \; G(x+i0)-G(x-i0) = -2\pi i\rho(x)$$

- Quasi-momentum :

$$p(x) \equiv G(x) - \frac{1}{2x} \;\rightarrow\; p(x+i0)+p(x-i0) = 2\pi n_x$$

- Algebraic curve

$$\oint_{A_i} dp = 0, \quad \oint_{B_{ij}} dp = 2\pi(n_i - n_j), \quad \oint_{A_i} p(x)dx = 2\pi i \frac{M_i}{L}$$



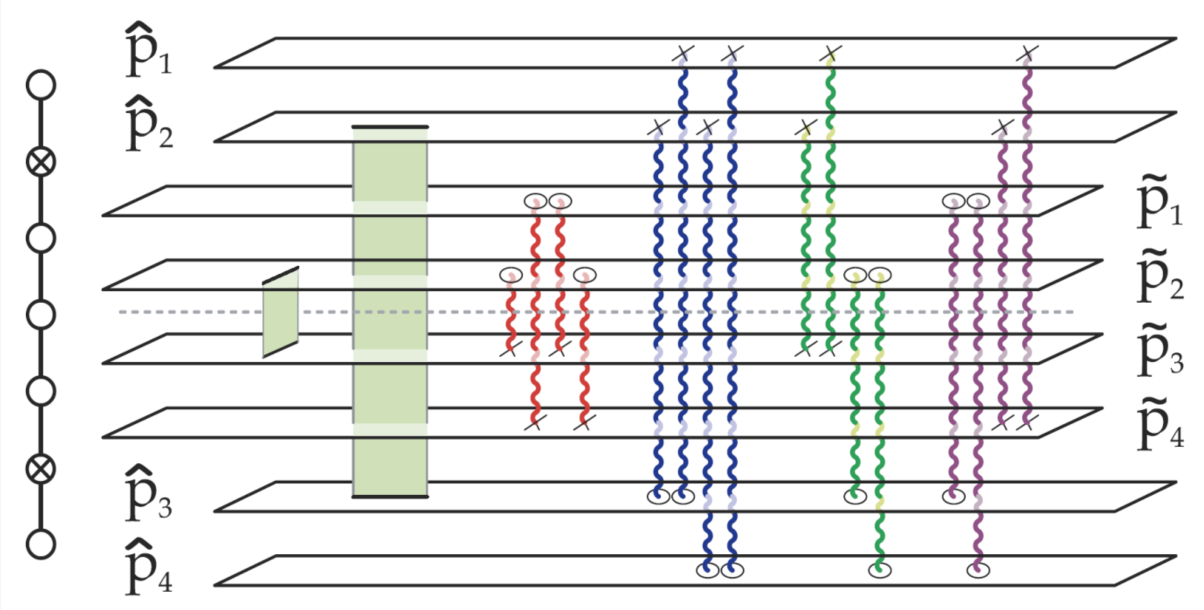
Algebraic curves of AdS/CFT

- BAE leads to the following algebraic structure
  - Eight quasi-momenta $(\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4 | \tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \tilde{p}_4)$
  - Discontinuities across the branch cuts

$$p_i(x + i0) - p_j(x - i0) = 2\pi n_{ij}\,, \qquad x \in \mathcal{C}_n^{ij}$$

$$i \in \{\tilde{1}, \tilde{2}, \widehat{2}, \widehat{2}\}\,, \qquad j \in \{\tilde{3}, \tilde{4}, \widehat{3}, \widehat{4}\}$$

$$\begin{aligned}
S^5 :&\quad (\tilde{1},\tilde{3})\,,(\tilde{1},\tilde{4})\,,(\tilde{2},\tilde{3})\,,(\tilde{2},\tilde{4})\\
AdS_5 :&\quad (\hat{1},\hat{3})\,,(\hat{1},\hat{4})\,,(\hat{2},\hat{3})\,,(\hat{2},\hat{4})\\
\text{Fermions} :&\quad (\tilde{1},\hat{3})\,,(\tilde{1},\hat{4})\,,(\tilde{2},\hat{3})\,,(\tilde{2},\hat{4})\\
&\quad (\hat{1},\tilde{3})\,,(\hat{1},\tilde{4})\,,(\hat{2},\tilde{3})\,,(\hat{2},\tilde{4})\,.
\end{aligned}$$



- Properties of quasi-momenta
  - Virasoro constraint :

$$\{\hat{p}_1,\hat{p}_2,\hat{p}_3,\hat{p}_4|\tilde{p}_1,\tilde{p}_2,\tilde{p}_3,\tilde{p}_4\} = \frac{\{\alpha_\pm,\alpha_\pm,\beta_\pm,\beta_\pm|\alpha_\pm,\alpha_\pm,\beta_\pm,\beta_\pm\}}{x\pm 1} + \mathcal{O}(1)$$

  - Conserved charges

$$\left(\begin{array}{c} \hat{p}_1 \\ \hat{p}_2 \\ \hat{p}_3 \\ \hat{p}_4 \\ \hline \tilde{p}_1 \\ \tilde{p}_2 \\ \tilde{p}_3 \\ \tilde{p}_4 \end{array}\right) = \frac{2\pi}{x}\left(\begin{array}{c} +\mathcal{E}-\mathcal{S}_1+\mathcal{S}_2 \\ +\mathcal{E}+\mathcal{S}_1-\mathcal{S}_2 \\ -\mathcal{E}-\mathcal{S}_1-\mathcal{S}_2 \\ -\mathcal{E}+\mathcal{S}_1+\mathcal{S}_2 \\ \hline +\mathcal{J}_1+\mathcal{J}_2-\mathcal{J}_3 \\ +\mathcal{J}_1-\mathcal{J}_2+\mathcal{J}_3 \\ -\mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3 \\ -\mathcal{J}_1-\mathcal{J}_2-\mathcal{J}_3 \end{array}\right) + \mathcal{O}\left(\frac{1}{x^2}\right)$$

$$E = \frac{\sqrt{\lambda}}{4\pi}\lim_{x\to\infty} x(\hat{p}_1(x)+\hat{p}_2(x))$$

  - Inversion relation from automorphism of psu(2,2|4)

$$\tilde{p}_{1,2}(x) = -\tilde{p}_{2,1}(1/x) - 2\pi m$$
$$\tilde{p}_{3,4}(x) = -\tilde{p}_{4,3}(1/x) + 2\pi m$$
$$\hat{p}_{1,2,3,4}(x) = -\hat{p}_{2,1,4,3}(1/x)$$

  - Filling fraction

$$S_{ij} = \pm\frac{\sqrt{\lambda}}{8\pi^2 i}\oint_{\mathcal{C}_{ij}}\left(1-\frac{1}{x^2}\right)p_i(x)dx$$

Lecture 2. Nonperturbative integrability
S-matrix

Plan

1. All-loop conjecture
2. Spin-chain S-matrix
3. World-sheet S-matrix
4. Symmetries of AdS/CFT
5. S-matrix of AdS/CFT
6. Dressing factor
7. Asymptotic Bethe ansatz

Fundamental questions

- Why does it work?
- Derivation rather than guess?
- Eigenvalues and Bethe ansatz for unknown spin chain Hamiltonian?
- What is the picture from the string theory?

S-matrix program

Symmetries

S-matrix

Wrapping Problem

All-loop BAE

Quantization of String theory

Perturbative spin-chain S-matrix

su(2) spin-chain S-matrix

- XXX Hamiltonian $$H = \sum_{l=1}^{L} \left[1 - \mathbf{P}_{l,l+1} \right]$$

- 2-magnon states

$$|\psi(p_1,p_2)\rangle = A_{XX}(12)|X(p_1)X(p_2)\rangle + A_{XX}(21)|X(p_2)X(p_1)\rangle,$$

$$|X(p_i)X(p_j)\rangle = \sum_{n_1<n_2} e^{i(p_i n_1 + p_j n_2)}|\overset{\overset{1}{\downarrow}}{Z}\cdots\overset{\overset{n_1}{\downarrow}}{X}\cdots\overset{\overset{n_2}{\downarrow}}{X}\cdots\overset{\overset{L}{\downarrow}}{Z}\rangle.$$

satisfy $$H|\psi\rangle = E(p_1,p_2)|\psi\rangle = \left(4\sin^2\frac{p_1}{2} + 4\sin^2\frac{p_2}{2}\right)|\psi\rangle$$

if $A_{XX}(21) = S(p_2,p_1)A_{XX}(12)$ with $$\boxed{S(p_2,p_1) = \frac{u_2 - u_1 + i}{u_2 - u_1 - i}}$$

↑

One-loop X-X scattering amplitude

su(3) spin-chain S-matrix

- Hamiltonian

$$H = \sum_{l=1}^{L} \left[1 - \mathbf{P}_{l,l+1} \right]$$

- 2-magnon states

$$|\psi\rangle = A_{XY}(12)|X(p_1)Y(p_2)\rangle + A_{XY}(21)|X(p_2)Y(p_1)\rangle + A_{YX}(12)|Y(p_1)X(p_2)\rangle + A_{YX}(21)|Y(p_2)X(p_1)\rangle$$

$$|\phi_1(p_i)\phi_2(p_j)\rangle = \sum_{n_1<n_2} e^{i(p_i n_1 + p_j n_2)} | \overset{\overset{1}{\downarrow}}{Z} \cdots \overset{\overset{n_1}{\downarrow}}{\phi_1} \cdots \overset{\overset{n_2}{\downarrow}}{\phi_2} \cdots \overset{\overset{L}{\downarrow}}{Z} \rangle$$

satisfy $$H|\psi\rangle = E(p_1,p_2)|\psi\rangle = \left(4\sin^2\frac{p_1}{2} + 4\sin^2\frac{p_2}{2} \right)|\psi\rangle$$

One-loop X-Y scattering amplitude

if $$\begin{pmatrix} A_{XY}(21) \\ A_{YX}(21) \end{pmatrix} = \boxed{\begin{pmatrix} R(p_2,p_1) & T(p_2,p_1) \\ T(p_2,p_1) & R(p_2,p_1) \end{pmatrix}} \begin{pmatrix} A_{XY}(12) \\ A_{YX}(12) \end{pmatrix}$$

$$T(p_2,p_1) = \frac{u_2 - u_1}{u_2 - u_1 - i}, \qquad R(p_2,p_1) = \frac{i}{u_2 - u_1 - i}$$

- S-matrix

$$\mathbf{S} = \begin{pmatrix} S & & & \\ & T & R & \\ & R & T & \\ & & & S \end{pmatrix} \propto \begin{pmatrix} u+i & & & \\ & u & i & \\ & i & u & \\ & & & u+i \end{pmatrix} \leftarrow$$

su(2) R-matrix
symmetry of
magnons

so(6) spin-chain S-matrix

- Hamiltonian $H = \sum_{l=1}^{L} \left(1 - \mathbf{P}_{l,l+1} + \frac{1}{2}\mathbf{K}_{l,l+1}\right)$ $\quad \mathbf{K}\,\Phi_i \otimes \Phi_j = \delta_{ij}\left(\sum_{k=1}^{6} \Phi_k \otimes \Phi_k\right)$
- Same as su(3) except the cases like $|X(p_2)\bar{X}(p_1)\rangle$

$$|\psi\rangle = \sum_{\phi=X,Y} \left[A_{\phi\bar{\phi}}(12)|\phi(p_1)\bar{\phi}(p_2)\rangle + A_{\phi\bar{\phi}}(21)|\phi(p_2)\bar{\phi}(p_1)\rangle + A_{\bar{\phi}\phi}(12)|\bar{\phi}(p_1)\phi(p_2)\rangle + A_{\bar{\phi}\phi}(21)|\bar{\phi}(p_1)\phi(p_1)\rangle\right]$$
$$+A_{\bar{Z}}|\bar{Z}(p_1+p_2)\rangle$$

$$|\phi(p_i)\bar{\phi}(p_j)\rangle = \sum_{n_1<n_2} e^{i(p_i n_1 + p_j n_2)}\,|\overset{\substack{1\\\downarrow}}{Z}\cdots\overset{\substack{n_1\\\downarrow}}{\phi}\cdots\overset{\substack{n_2\\\downarrow}}{\bar{\phi}}\cdots\overset{\substack{L\\\downarrow}}{Z}\rangle \qquad |\bar{Z}(p)\rangle = \sum_{n} e^{ipn}|\overset{\substack{1\\\downarrow}}{Z}\cdots\overset{\substack{n\\\downarrow}}{\bar{Z}}\cdots\overset{\substack{L\\\downarrow}}{Z}\rangle$$

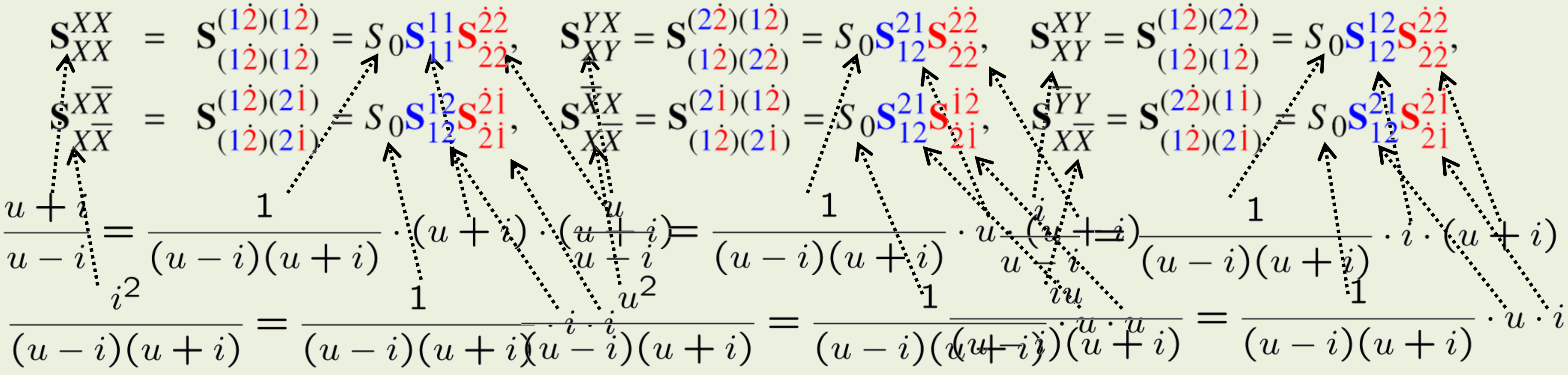
satisfy $H|\psi\rangle = E(p_1,p_2)|\psi\rangle = \left(4\sin^2\frac{p_1}{2} + 4\sin^2\frac{p_2}{2}\right)|\psi\rangle$

if
$$\begin{pmatrix} A_{X\bar{X}}(21) \\ A_{\bar{X}X}(21) \\ A_{Y\bar{Y}}(21) \\ A_{\bar{Y}Y}(21) \end{pmatrix} = \boxed{\begin{pmatrix} \mathcal{R}(p_2,p_1) & \mathcal{T}(p_2,p_1) & \mathcal{S}(p_2,p_1) & \mathcal{S}(p_2,p_1) \\ \mathcal{T}(p_2,p_1) & \mathcal{R}(p_2,p_1) & \mathcal{S}(p_2,p_1) & \mathcal{S}(p_2,p_1) \\ \mathcal{S}(p_2,p_1) & \mathcal{S}(p_2,p_1) & \mathcal{R}(p_2,p_1) & \mathcal{T}(p_2,p_1) \\ \mathcal{S}(p_2,p_1) & \mathcal{S}(p_2,p_1) & \mathcal{T}(p_2,p_1) & \mathcal{R}(p_2,p_1) \end{pmatrix}} \begin{pmatrix} A_{X\bar{X}}(12) \\ A_{\bar{X}X}(12) \\ A_{Y\bar{Y}}(12) \\ A_{\bar{Y}Y}(12) \end{pmatrix}$$

$$\mathcal{T}(p_2,p_1) = \frac{(u_2-u_1)^2}{(u_2-u_1-i)(u_2-u_1+i)},\ \mathcal{R}(p_2,p_1) = \frac{-1}{(u_2-u_1-i)(u_2-u_1+i)},\ \mathcal{S}(p_2,p_1) = \frac{i(u_2-u_1)}{(u_2-u_1-i)(u_2-u_1+i)}$$

One-loop so(6) scattering amplitude

- so(6) S-matrix can be factorized into a tensor product !
- Define $X = 1\dot{2}$, $\bar{X} = 2\dot{1}$, $Y = 2\dot{2}$, $\bar{Y} = 1\dot{1}$ $(u \equiv u_2 - u_1)$



$$\mathbf{S}^{XX}_{XX} = \mathbf{S}^{(1\dot{2})(1\dot{2})}_{(1\dot{2})(1\dot{2})} = S_0 \mathbf{S}^{11}_{11} \mathbf{S}^{\dot{2}\dot{2}}_{\dot{2}\dot{2}}, \quad \mathbf{S}^{YX}_{XY} = \mathbf{S}^{(2\dot{2})(1\dot{2})}_{(1\dot{2})(2\dot{2})} = S_0 \mathbf{S}^{21}_{12} \mathbf{S}^{\dot{2}\dot{2}}_{\dot{2}\dot{2}}, \quad \mathbf{S}^{XY}_{XY} = \mathbf{S}^{(1\dot{2})(2\dot{2})}_{(1\dot{2})(1\dot{2})} = S_0 \mathbf{S}^{12}_{12} \mathbf{S}^{\dot{2}\dot{2}}_{\dot{2}\dot{2}},$$

$$\mathbf{S}^{X\bar{X}}_{X\bar{X}} = \mathbf{S}^{(1\dot{2})(2\dot{1})}_{(1\dot{2})(2\dot{1})} = S_0 \mathbf{S}^{12}_{12} \mathbf{S}^{\dot{2}\dot{1}}_{\dot{2}\dot{1}}, \quad \mathbf{S}^{\bar{X}X}_{X\bar{X}} = \mathbf{S}^{(2\dot{1})(1\dot{2})}_{(1\dot{2})(2\dot{1})} = S_0 \mathbf{S}^{21}_{12} \mathbf{S}^{\dot{1}\dot{2}}_{\dot{2}\dot{1}}, \quad \mathbf{S}^{\bar{Y}Y}_{X\bar{X}} = \mathbf{S}^{(2\dot{2})(1\dot{1})}_{(1\dot{2})(2\dot{1})} = S_0 \mathbf{S}^{21}_{12} \mathbf{S}^{\dot{2}\dot{1}}_{\dot{2}\dot{1}}$$

$$\frac{u+i}{u-i} = \frac{1}{(u-i)(u+i)} \cdot (u+i) \cdot (u+i) \qquad \frac{u}{u-i} = \frac{1}{(u-i)(u+i)} \cdot u \cdot (u+i) \qquad \frac{i}{u-i} = \frac{1}{(u-i)(u+i)} \cdot i \cdot (u+i)$$

$$\frac{i^2}{(u-i)(u+i)} = \frac{1}{(u-i)(u+i)} \cdot i \cdot i \qquad \frac{u^2}{(u-i)(u+i)} = \frac{1}{(u-i)(u+i)} \cdot u \cdot u \qquad \frac{iu}{(u-i)(u+i)} = \frac{1}{(u-i)(u+i)} \cdot u \cdot i$$

$$\mathbf{S}_{so(6)} = S_0 \cdot \mathbf{S} \otimes \mathbf{S}, \qquad S_0 = (u^2+1)^{-1}$$

$$\mathbf{S} = \mathbf{S} = \begin{pmatrix} u+i & & & \\ & u & i & \\ & i & u & \\ & & & u+i \end{pmatrix}$$

$$SO(6) \to SO(4) \simeq SU(2) \times SU(2)$$

Worldsheet S-matrix

- So far we considered S-matrix from gauge theory spin chains
- String perturbative computation (large λ) of S−matrix is also possible
  - Fluctuation around BMN in light-cone gauge
  - Effective Lagrangian contains
    - Quadratic terms in terms of oscillator algebra [BMN limit]
    - Quartic interaction terms

$$\mathcal{L}_{\text{int.}} \sim \frac{1}{\sqrt{\lambda}}\left[x^2(p_y^2 + y'^2) - y^2(p_x^2 + x'^2) + 2x^2x'^2 - 2y^2y'^2\right]$$

- Can compute scattering amplitudes on the worldsheet

Klose,McLoughlin,Roiban,Zarembo (2007)

Beyond perturbation

Excitation spectrum

- Scalar sector

$$su(3)\ :\ \{Z,X,Y\}\quad\rightarrow\quad su(2)\ :\ \mathrm{Tr}\,[Z\cdots ZXZ\cdots ZY\cdots Z]$$

$$so(6)\ :\ \{Z,X,Y,\overline{Z},\overline{X},\overline{Y}\}\quad\rightarrow\quad so(4)\simeq su(2)_a\times su(2)_{\dot a}\ :\ \mathrm{Tr}\left[Z\cdots XZ\overline{X}Z\cdots ZY\cdots Z\overline{Y}\right]$$

- Fermion sector $su(2,2)\supset su(2)_\alpha\times su(2)_{\dot\alpha}\quad a,\dot a=1,\ 2\ ;\alpha,\dot\alpha=3,\ 4$
- Full sector $\mathrm{Tr}\,[Z\cdots\chi_1Z\cdots\chi_2Z\cdots\chi_3Z\cdots Z\chi_nZ\cdots]$

Any field except Z, $\overline{\mathrm{Z}}$

	$so(6)_R$
A_μ	$\mathbf{1}$
χ^A_α $\bar\chi^{\bar A}_{\bar\alpha}$	$\mathbf{4}\oplus\bar{\mathbf{4}}$
Φ^a	$\cancel{\mathbf{6}}\,\mathbf{4}$

$$\Phi_{a\dot a}=\phi_a\phi_{\dot a},\quad \chi^a_{\dot\alpha}=\phi_a\psi_{\dot\alpha},\quad \chi^{\dot a}_{\alpha}=\psi_\alpha\phi_{\dot a},\quad \mathcal{D}_{\alpha\dot\alpha}=\psi_\alpha\psi_{\dot\alpha}$$

  - Excitations of N=4 SYM:

$$(\boxslash;\boxslash)=\Phi_{a\dot a}\oplus\chi^a_{\dot\alpha}\oplus\chi^{\dot a}_{\alpha}\oplus D_{\alpha\dot\alpha}$$

  - Each SYM field is a “meson” made of a “quark” and an “anti-quark”

$$\boxslash=(\phi_a|\psi_\alpha)=(\phi_1,\phi_2|\psi_3,\psi_4)$$

Centrally extended su(2|2) symmetry

- Symmetry of the excitations: su(2|2) x su(2|2) **Beisert (2008)**

$$\left(\begin{array}{c|c} \mathbb{L}_a{}^b & \mathbb{Q}_\alpha{}^b \\ \hline \mathbb{Q}_a^{\dagger\beta} & \mathbb{R}_\alpha{}^\beta \end{array}\right), \quad \left(\begin{array}{c|c} \mathbb{L}_{\dot a}{}^{\dot b} & \mathbb{Q}_{\dot\alpha}{}^{\dot b} \\ \hline \mathbb{Q}_{\dot a}^{\dagger\dot\beta} & \mathbb{R}_{\dot\alpha}{}^{\dot\beta} \end{array}\right)$$

- Fundamental representation $\boxslash = (\phi_a|\psi_\alpha) = (\phi_1, \phi_2|\psi_3, \psi_4)$
- Commutation relations

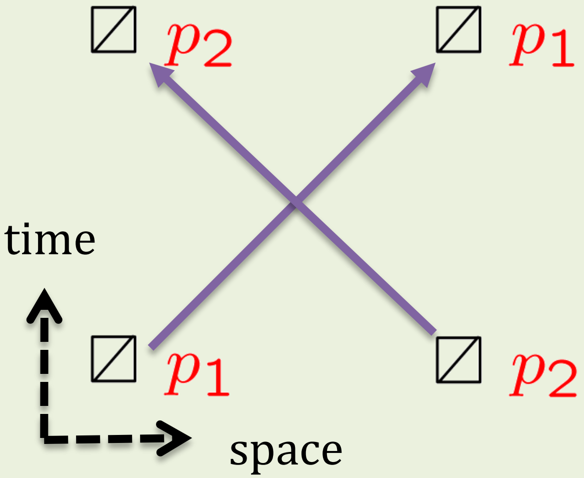
$$\begin{aligned}
\left[\mathbb{L}_a{}^b, \mathbb{J}_c\right] &= \delta_c^b \mathbb{J}_a - \frac{1}{2}\delta_a^b \mathbb{J}_c\,, \quad \left[\mathbb{R}_\alpha{}^\beta, \mathbb{J}_\gamma\right] = \delta_\gamma^\beta \mathbb{J}_\alpha - \frac{1}{2}\delta_\alpha^\beta \mathbb{J}_\gamma\,, \\
\left[\mathbb{L}_a{}^b, \mathbb{J}^c\right] &= -\delta_a^c \mathbb{J}^b + \frac{1}{2}\delta_a^b \mathbb{J}^c\,, \quad \left[\mathbb{R}_\alpha{}^\beta, \mathbb{J}^\gamma\right] = -\delta_\alpha^\gamma \mathbb{J}^\beta + \frac{1}{2}\delta_\alpha^\beta \mathbb{J}^\gamma\,, \\
\{\mathbb{Q}_\alpha{}^a, \mathbb{Q}_\beta{}^b\} &= \epsilon_{\alpha\beta}\epsilon^{ab}\mathbb{C}\,, \quad \{\mathbb{Q}_a^{\dagger\alpha}, \mathbb{Q}_b^{\dagger\beta}\} = \epsilon^{\alpha\beta}\epsilon_{ab}\mathbb{C}^\dagger\,, \\
\{\mathbb{Q}_\alpha{}^a, \mathbb{Q}_b^{\dagger\beta}\} &= \delta_b^a \mathbb{R}_\alpha{}^\beta + \delta_\alpha^\beta \mathbb{L}_b{}^a + \frac{1}{2}\delta_b^a\delta_\alpha^\beta \mathbb{H}
\end{aligned}$$

S-matrix from su(2|2) symmetry

- S-matrix $\mathbb{S} = \color{blue}{\mathbf{S}} \otimes \color{red}{\dot{\mathbf{S}}}$, $\quad \color{blue}{\mathbf{S}} = \color{red}{\dot{\mathbf{S}}}$
- Focus only one from now on
- S-matrix should commute with su(2|2)

$$\left[\color{blue}{\mathbf{S}}(p_1, p_2), \left(\begin{array}{c|c} \mathbb{L}_a{}^b & \mathbb{Q}_\alpha{}^b \\ \hline \mathbb{Q}^{\dagger\beta}_a & \mathbb{R}_\alpha{}^\beta \end{array} \right) \right] = 0$$

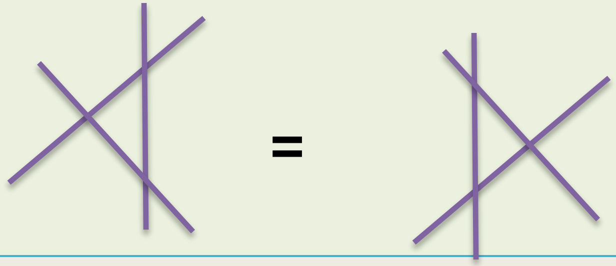
Beisert (2008)



- Reformulate as algebraic problem
- Yang-Baxter equation

Arutyunov, Frolov, Zamaklar (2008)

$$\mathbf{S}_{12}(p_1, p_2)\,\mathbf{S}_{13}(p_1, p_3)\,\mathbf{S}_{23}(p_2, p_3) = \mathbf{S}_{23}(p_2, p_3)\,\mathbf{S}_{13}(p_1, p_3)\,\mathbf{S}_{12}(p_1, p_2)$$



Zamolodchikov-Faddeev algebra

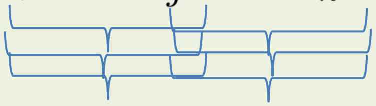
- Two-particle state $|\phi_i(p_1)\phi_j(p_2)\rangle = \mathcal{A}_i^\dagger(p_1)\mathcal{A}_j^\dagger(p_2)|0\rangle$ $\quad i = (a,\alpha) = \underbrace{1,2}_{\text{Latin}},\underbrace{3,4}_{\text{Greek}}$

- S-matrix defines ZF algebra

$$\mathcal{A}_i^\dagger(p_1)\,\mathcal{A}_j^\dagger(p_2) = S_{i\,j}^{i'j'}(p_1,p_2)\,\mathcal{A}_{j'}^\dagger(p_2)\,\mathcal{A}_{i'}^\dagger(p_1)$$

- Yang-Baxter equation appears from associativity

$$\mathcal{A}_i^\dagger(p_1)\mathcal{A}_j^\dagger(p_2)\mathcal{A}_k^\dagger(p_3) \quad \rightarrow \quad \mathcal{A}_{k'}^\dagger(p_3)\mathcal{A}_{j'}^\dagger(p_2)\mathcal{A}_{i'}^\dagger(p_1)$$



- Acting bosonic su(2|2) generators on ZF generators

$$\begin{aligned}\left[\mathbb{L}_a{}^b, \mathcal{A}_c^\dagger(p)\right] &= (\delta_c^b\delta_a^d - \frac{1}{2}\delta_a^b\delta_c^d)\mathcal{A}_d^\dagger(p)\,, \quad \left[\mathbb{L}_a{}^b, \mathcal{A}_\gamma^\dagger(p)\right] = 0\,,\\ \left[\mathbb{R}_\alpha{}^\beta, \mathcal{A}_\gamma^\dagger(p)\right] &= (\delta_\gamma^\beta\delta_\alpha^\delta - \frac{1}{2}\delta_\alpha^\beta\delta_\gamma^\delta)\mathcal{A}_\delta^\dagger(p)\,, \quad \left[\mathbb{R}_\alpha{}^\beta, \mathcal{A}_c^\dagger(p)\right] = 0\end{aligned}$$

- Acting fermionic su(2|2) generators on ZF generators

$$\begin{aligned}
\mathbb{Q}_\alpha{}^a\,\mathcal{A}_b^\dagger(p) &= e^{-ip/2}\left[a(p)\delta_b^a\mathcal{A}_\alpha^\dagger(p)+\mathcal{A}_b^\dagger(p)\,\mathbb{Q}_\alpha{}^a\right],\\
\mathbb{Q}_\alpha{}^a\,\mathcal{A}_\beta^\dagger(p) &= e^{-ip/2}\left[b(p)\epsilon_{\alpha\beta}\epsilon^{ab}\mathcal{A}_b^\dagger(p)-\mathcal{A}_\beta^\dagger(p)\,\mathbb{Q}_\alpha{}^a\right],\\
\mathbb{Q}_a^{\dagger\alpha}\,\mathcal{A}_b^\dagger(p) &= e^{ip/2}\left[c(p)\epsilon_{ab}\epsilon^{\alpha\beta}\mathcal{A}_\beta^\dagger(p)+\mathcal{A}_b^\dagger(p)\,\mathbb{Q}_a^{\dagger\alpha}\right],\\
\mathbb{Q}_a^{\dagger\alpha}\,\mathcal{A}_\beta^\dagger(p) &= e^{ip/2}\left[d(p)\delta_\beta^\alpha\mathcal{A}_a^\dagger(p)-\mathcal{A}_\beta^\dagger(p)\,\mathbb{Q}_a^{\dagger\alpha}\right]
\end{aligned}$$

- Central charges act on

$$\begin{aligned}
\mathbb{C}\,\mathcal{A}_i^\dagger(p) &= e^{-ip}\left[a(p)b(p)\mathcal{A}_i^\dagger(p)+\mathcal{A}_i^\dagger(p)\,\mathbb{C}\right],\\
\mathbb{C}^\dagger\,\mathcal{A}_i^\dagger(p) &= e^{ip}\left[c(p)d(p)\mathcal{A}_i^\dagger(p)+\mathcal{A}_i^\dagger(p)\,\mathbb{C}^\dagger\right],\\
\mathbb{H}\,\mathcal{A}_i^\dagger(p) &= \left[a(p)d(p)+b(p)c(p)\right]\mathcal{A}_i^\dagger(p)+\mathcal{A}_i^\dagger(p)\,\mathbb{H}
\end{aligned}$$

  - ZF generators form a su(2|2) representation if $ad-bc=1$
  - Unitary representation if $d=a^*,\quad c=b^*$

$$\{\mathbb{Q}_\alpha{}^a,\mathbb{Q}_b^{\dagger\beta}\}=\delta_b^a\mathbb{R}_\alpha{}^\beta+\delta_\alpha^\beta\mathbb{L}_b{}^a+\frac{1}{2}\delta_b^a\delta_\alpha^\beta\mathbb{H}$$

  - Acting $\mathbb{C}$ on two-particle scattering states

$$e^{-ip_1}a(p_1)b(p_1)+e^{-i(p_1+p_2)}a(p_2)b(p_2)=e^{-ip_2}a(p_2)b(p_2)+e^{-i(p_1+p_2)}a(p_1)b(p_1)$$

$$a(p)b(p)=ig(e^{ip}-1)$$

some constant

- From these one can determine the parameters

$$a = \sqrt{g}\eta\,, \quad b = \sqrt{g}\frac{i}{\eta}\left(\frac{x^+}{x^-} - 1\right)\,, \quad c = -\sqrt{g}\frac{\eta}{x^+}\,, \quad d = \sqrt{g}\frac{x^+}{i\eta}\left(1 - \frac{x^-}{x^+}\right)$$

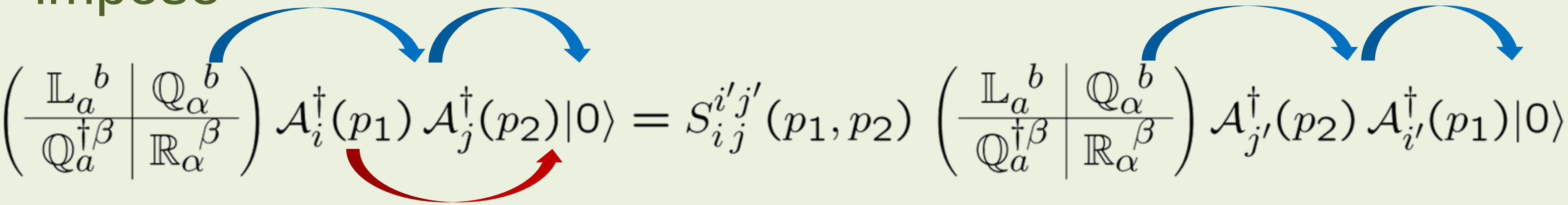
$$\boxed{x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{g}\,, \qquad \frac{x^+}{x^-} = e^{ip}}, \qquad \eta = e^{ip/4}\sqrt{i(x^- - x^+)}$$

$$x^\pm = e^{\pm i\frac{p}{2}}\left[\frac{1 + \sqrt{1 + 16g^2 \sin^2\frac{p}{2}}}{4g\sin\frac{p}{2}}\right]\,, \quad u = \frac{1}{2}\cot\frac{p}{2}\sqrt{1 + 16g^2\sin^2\frac{p}{2}}$$

- Central charge $\mathbb{H} = -ig\left(x^+ - \frac{1}{x^+} - x^- + \frac{1}{x^-}\right) = \sqrt{1 + 16g^2\sin^2\frac{p}{2}}$
  - Comparing with conformal dimension at weak coupling limit and BMN limit, one can conclude

$$g = g \equiv \frac{\sqrt{\lambda}}{4\pi}$$

- Act the generators on the two-particle scattering states and impose



$$\left(\begin{array}{c|c} \mathbb{L}_a{}^b & \mathbb{Q}_\alpha{}^b \\ \hline \mathbb{Q}^{\dagger\beta}_a & \mathbb{R}_\alpha{}^\beta \end{array}\right) \mathcal{A}^\dagger_i(p_1)\,\mathcal{A}^\dagger_j(p_2)|0\rangle = S_{i\,j}^{i'j'}(p_1,p_2)\left(\begin{array}{c|c} \mathbb{L}_a{}^b & \mathbb{Q}_\alpha{}^b \\ \hline \mathbb{Q}^{\dagger\beta}_a & \mathbb{R}_\alpha{}^\beta \end{array}\right) \mathcal{A}^\dagger_{j'}(p_2)\,\mathcal{A}^\dagger_{i'}(p_1)|0\rangle$$

  - Generates a set of linear coupled equations for the S-matrix elements and can be solved uniquely up to an overall function

- S : 16 x 16 matrix

$$\begin{array}{c|cc} & (ab)(\alpha\beta) & (a\beta)(\alpha b) \\ \hline (ab)(\alpha\beta) & \blacksquare & 0 \\ (a\beta)(\alpha b) & 0 & \blacksquare \end{array}$$

$$\begin{aligned}
S^{a\,a}_{a\,a} &= A\,, \quad S^{\alpha\,\alpha}_{\alpha\,\alpha} = D\,,\\
S^{a\,b}_{a\,b} &= \frac{1}{2}(A-B)\,, \quad S^{b\,a}_{a\,b} = \frac{1}{2}(A+B)\,,\\
S^{\alpha\,\beta}_{\alpha\,\beta} &= \frac{1}{2}(D-E)\,, \quad S^{\beta\,\alpha}_{\alpha\,\beta} = \frac{1}{2}(D+E)\,,\\
S^{\alpha\,\beta}_{a\,b} &= -\frac{1}{2}\epsilon_{ab}\epsilon^{\alpha\beta}C\,, \quad S^{a\,b}_{\alpha\,\beta} = -\frac{1}{2}\epsilon^{ab}\epsilon_{\alpha\beta}F\,,\\
S^{a\,\alpha}_{a\,\alpha} &= G\,, \quad S^{\alpha\,a}_{a\,\alpha} = H\,, \quad S^{a\,\alpha}_{\alpha\,a} = K\,, \quad S^{\alpha\,a}_{\alpha\,a} = L
\end{aligned}$$

$$\begin{aligned}
A &= S_0\frac{x_2^- - x_1^+}{x_2^+ - x_1^-}\frac{\eta_1\eta_2}{\tilde\eta_1\tilde\eta_2}\,,\\
B &= -S_0\left[\frac{x_2^- - x_1^+}{x_2^+ - x_1^-} + 2\frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_2^- + x_1^+)}{(x_1^- - x_2^+)(x_1^-x_2^- - x_1^+x_2^+)}\right]\frac{\eta_1\eta_2}{\tilde\eta_1\tilde\eta_2}\,,\\
C &= S_0\frac{2ix_1^-x_2^-(x_1^+ - x_2^+)\eta_1\eta_2}{x_1^+x_2^+(x_1^- - x_2^+)(1 - x_1^-x_2^-)}\,, \qquad D = -S_0\,,\\
E &= S_0\left[1 - 2\frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^- + x_2^+)}{(x_1^- - x_2^+)(x_1^-x_2^- - x_1^+x_2^+)}\right]\,,\\
F &= S_0\frac{2i(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^+ - x_2^+)}{(x_1^- - x_2^+)(1 - x_1^-x_2^-)\tilde\eta_1\tilde\eta_2}\,,\\
G &= S_0\frac{(x_2^- - x_1^-)}{(x_2^+ - x_1^-)}\frac{\eta_1}{\tilde\eta_1}\,, \qquad H = S_0\frac{(x_2^+ - x_2^-)}{(x_1^- - x_2^+)}\frac{\eta_1}{\tilde\eta_2}\,,\\
K &= S_0\frac{(x_1^+ - x_1^-)}{(x_1^- - x_2^+)}\frac{\eta_2}{\tilde\eta_1}\,, \qquad L = S_0\frac{(x_1^+ - x_2^+)}{(x_1^- - x_2^+)}\frac{\eta_2}{\tilde\eta_2}
\end{aligned}$$

$$\eta_1 = \eta(p_1)e^{ip_2/2}\,, \quad \eta_2 = \eta(p_2)\,, \quad \tilde\eta_1 = \eta(p_1)\,, \quad \tilde\eta_2 = \eta(p_2)e^{ip_1/2}$$

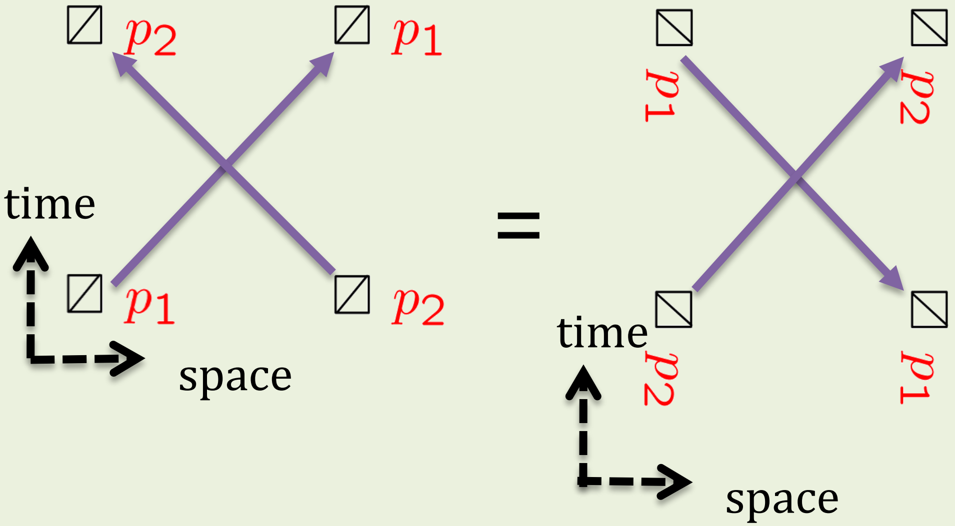
su(2|2) S-matrix

$$\left(\begin{array}{cccc|cccc|cccc|cccc}
a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a_{10} & 0 & 0 & a_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a_{10} & 0 & 0 & 0 & 0 & 0 & a_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -a_2 & 0 & 0 & -a_7 & 0 & 0 & a_7 & 0 & 0 & a_1+a_2 & 0 & 0 & 0 \\
\hline
0 & a_5 & 0 & 0 & a_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -a_8 & 0 & 0 & -a_4 & 0 & 0 & a_3+a_4 & 0 & 0 & a_8 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_9 & 0 & 0 & 0 & 0 & 0 & a_5 & 0 & 0 \\
\hline
0 & 0 & a_5 & 0 & 0 & 0 & 0 & 0 & a_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_8 & 0 & 0 & a_3+a_4 & 0 & 0 & -a_4 & 0 & 0 & -a_8 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_9 & 0 & 0 & a_5 & 0 \\
\hline
0 & 0 & 0 & a_1+a_2 & 0 & 0 & a_7 & 0 & 0 & -a_7 & 0 & 0 & -a_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_6 & 0 & 0 & 0 & 0 & 0 & a_{10} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_6 & 0 & 0 & a_{10} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1
\end{array}\right)$$

R-matrix of Hubbard model

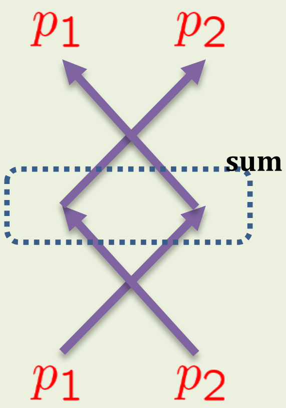
Dressing factor

- YBE, symmetry DO NOT determine the overall function
  - Crossing and unitarity along with bound state spectrum
- Crossing symmetry from space←→time



- Unitarity :

$$\mathbf{S}(p_1,p_2)\cdot\mathbf{S}(p_2,p_1)=\mathbf{I}$$



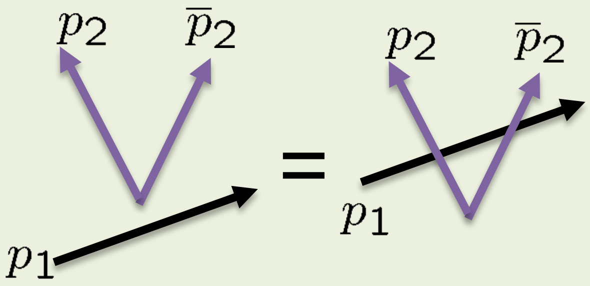
- ## Crossing-unitarity from a singlet operator

$$x^{\pm}(\bar{p}) = \frac{1}{x^{\pm}(p)}$$

$$\mathbb{H},\ p \ \to\ -\mathbb{H},\ -p$$

$$I(p) = C^{ij}(p)\, A_i^\dagger(p)\, A_j^\dagger(\bar{p}) \equiv -i\epsilon^{ab} A_a^\dagger(p)\, A_b^\dagger(\bar{p}) + \epsilon^{\alpha\beta} A_\alpha^\dagger(p)\, A_\beta^\dagger(\bar{p})$$

Charge conjugation



$$\begin{aligned} A_i^\dagger(p_1)\, I(p_2) &= C^{jk}(p_2)\, A_i^\dagger(p_1)\, A_j^\dagger(p_2)\, A_k^\dagger(\bar{p}_2) \\ &= C^{jk}(p_2)\, S_{ij}^{i'j'}(p_1,p_2)\, A_{j'}^\dagger(p_2)\, A_{i'}^\dagger(p_1)\, A_k^\dagger(\bar{p}_2) \\ &= \boxed{C^{jk}(p_2)\, S_{ij}^{i'j'}(p_1,p_2)\, S_{i'k}^{i''k'}(p_1,\bar{p}_2)}\, A_{j'}^\dagger(p_2)\, A_{k'}^\dagger(\bar{p}_2)\, A_{i''}^\dagger(p_1) \\ &\equiv I(p_2)\, A_i^\dagger(p_1) \qquad\qquad \propto C^{j'k'}\delta_i^{j''} \end{aligned}$$

$$S_0(p_1,p_2)\, S_0(p_1,\bar{p}_2) = \frac{\left(\frac{1}{x_1^-} - x_2^-\right)(x_1^- - x_2^+)}{\left(\frac{1}{x_1^+} - x_2^-\right)(x_1^+ - x_2^+)}$$

$$S_0(p_1\,,p_2)^2 \equiv \frac{x_1^- - x_2^+}{x_1^+ - x_2^-}\frac{1 - \frac{1}{x_1^+ x_2^-}}{1 - \frac{1}{x_1^- x_2^+}}\sigma(p_1\,,p_2)^2$$

$$\sigma(p_1,p_2)\,\sigma(\bar{p}_1,p_2) = \frac{1 - \frac{1}{x_1^+ x_2^+}}{1 - \frac{x_1^-}{x_2^-}} \frac{1 - \frac{x_1^-}{x_2^+}}{1 - \frac{1}{x_1^+ x_2^-}}$$

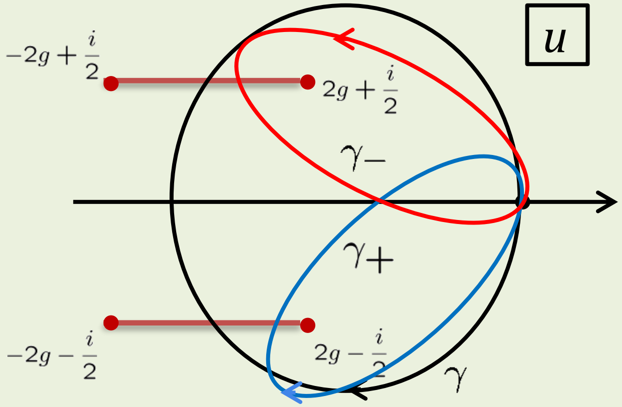
Janik (2006)

- ## Zhukovsky map [rescaled $u \to \frac{u}{g}$]

$$x + \frac{1}{x} = \frac{u}{g}, \qquad x^{\pm} + \frac{1}{x^{\pm}} = \frac{1}{g}\left(u \pm \frac{i}{2}\right)$$

- For real u : $|x^{\pm}| > 1$
- Branch cuts occur when
- Crossing cuts :

$$\begin{aligned} \gamma \quad &: \quad x^{\pm} \to \frac{1}{x^{\pm}} \\ \gamma_{-} \quad &: \quad x^{-} \to \frac{1}{x^{-}} \\ \gamma_{+} \quad &: \quad x^{+} \to \frac{1}{x^{+}} \end{aligned}$$



- Janik relation can be written as

$$\sigma(x,y)\,\sigma^{\gamma}(x,y) = \frac{1-\frac{1}{x^+y^+}}{1-\frac{x^-}{y^-}}\,\frac{1-\frac{x^-}{y^+}}{1-\frac{1}{x^+y^-}} \qquad x^{\pm} = x\left(u \pm \frac{i}{2}\right),\ y^{\pm} = x\left(v \pm \frac{i}{2}\right)$$

- Apply γ_ contour

$$\sigma^{\gamma_-}(x,y)\,\sigma^{\gamma_+}(x,y) = \frac{1-\frac{1}{x^+y^+}}{1-\frac{1}{x^-y^-}}\,\frac{1-\frac{1}{x^-y^+}}{1-\frac{1}{x^+y^-}} \qquad (*)$$

- Define a translation $D = e^{\frac{i}{2}\partial_u} \ :\ Df(u) = f(u+i/2) = e^{D\ln f} = f^{D}$

- RHS of (*)

$$\frac{1-\frac{1}{x^+y^+}}{1-\frac{1}{x^-y^-}}\,\frac{1-\frac{1}{x^-y^+}}{1-\frac{1}{x^+y^-}} = \frac{1-\frac{1}{x^+y^+}}{1-\frac{1}{x^+y^-}}\,\frac{1-\frac{1}{x^-y^+}}{1-\frac{1}{x^-y^-}} = \left(\frac{1-\frac{1}{xy^+}}{1-\frac{1}{xy^-}}\right)^{D}\left(\frac{1-\frac{1}{xy^+}}{1-\frac{1}{xy^-}}\right)^{D^{-1}} = \left(\frac{1-\frac{1}{xy^+}}{1-\frac{1}{xy^-}}\right)^{D+D^{-1}} = \left(\frac{x-\frac{1}{y^+}}{x-\frac{1}{y^-}}\right)^{D+D^{-1}}$$

- Define
$$\begin{aligned}\sigma(x,y) &= \mathrm{exp}\left\{i\left[\chi(x^+,y^-)+\chi(x^-,y^+)-\chi(x^+,y^+)-\chi(x^-,y^-)\right]\right\}\\ \sigma_1(x,y) &= \mathrm{exp}\left\{i\left[\chi(x,y^-)-\chi(x,y^+)\right]\right\}\end{aligned}$$

- Then,
$$\begin{aligned}\sigma^{\gamma-}(x,y) &= \mathrm{exp}\left\{i\left[\chi(x^+,y^-)+\chi(1/x^-,y^+)-\chi(x^+,y^+)-\chi(1/x^-,y^-)\right]\right\} = \frac{\sigma_1(x^+,y)}{\sigma_1(1/x^-,y)}\\ \sigma^{\gamma+}(x,y) &= \mathrm{exp}\left\{i\left[\chi(1/x^+,y^-)+\chi(x^-,y^+)-\chi(1/x^+,y^+)-\chi(x^-,y^-)\right]\right\} = \frac{\sigma_1(1/x^+,y)}{\sigma_1(x^-,y)}\end{aligned}$$

- LHS of (*)

$$\frac{\sigma_1(x^+,y)}{\sigma_1(x^-,y)}\,\frac{\sigma_1(1/x^+,y)}{\sigma_1(1/x^-,y)} = \frac{[\sigma_1(x,y)\sigma_1(1/x,y)]^D}{[\sigma_1(x,y)\sigma_1(1/x,y)]^{D^{-1}}} = [\sigma_1(x,y)\sigma_1(1/x,y)]^{D-D^{-1}} = \left(\frac{x-\frac{1}{y^+}}{x-\frac{1}{y^-}}\right)^{D+D^{-1}}$$

$$\sigma_1(x,y)\sigma_1(1/x,y) = \left(\frac{x-\frac{1}{y^+}}{x-\frac{1}{y^-}}\right)^{\frac{D+D^{-1}}{D-D^{-1}}}$$

$$\sigma_1(x,y)\sigma_1(1/x,y) = \mathrm{exp}\left\{i\left[\chi(x,y^-)-\chi(x,y^+)+\chi(1/x,y^-)-\chi(1/x,y^+)\right]\right\} = \frac{\mathrm{exp}\left\{i\left[\chi(x,y^-)+\chi(1/x,y^-)\right]\right\}}{\mathrm{exp}\left\{i\left[\chi(x,y^+)+\chi(1/x,y^+)\right]\right\}}$$

$$e^{i[\chi(x,y)+\chi(1/x,y)]} = \left(\frac{x-\frac{1}{y}}{\sqrt{x}}\right)^{-f(D)}, \qquad f(D) = \frac{D+D^{-1}}{D-D^{-1}}$$

$$e^{i[\chi(x,y)+\chi(1/x,y)+\chi(x,1/y)+\chi(1/x,1/y)]} = \left(\frac{x-\frac{1}{y}}{\sqrt{x}}\cdot\frac{x-y}{\sqrt{x}}\right)^{-f(D)} = \left(x+\frac{1}{x}-y-\frac{1}{y}\right)^{-f(D)} = (u-v)^{-f(D)}$$

- Using $f(D)=\frac{D+D^{-1}}{D-D^{-1}}=\frac{D^{-2}}{1-D^{-2}}-\frac{D^2}{1-D^2}=\sum_{n=1}^{\infty}D^{-2n}-\sum_{n=1}^{\infty}D^{2n}$

- One can find $(u-v)^{\sum_{n=1}^{\infty}D^{-2n}-\sum_{n=1}^{\infty}D^{2n}}=\prod_{n=1}^{\infty}\frac{u-v-in}{u-v+in}=\frac{\Gamma(1+iu-iv)}{\Gamma(1-iu+iv)}$

$$e^{i[\chi(x,y)+\chi(1/x,y)+\chi(x,1/y)+\chi(1/x,1/y)]}=\frac{\Gamma(1+iu-iv)}{\Gamma(1-iu+iv)}$$

- In terms of *u, v near the cuts*

$$\chi(u+i0,v+i0)+\chi(u-i0,v+i0)+\chi(u+i0,v-i0)+\chi(u-i0,v-i0)=\frac{1}{i}\ln\frac{\Gamma(1+iu-iv)}{\Gamma(1-iu+iv)}$$

- Riemann-Hilbert problem $\xi(u+i0)-\xi(u-i0)=F(u)\to\xi(u)=\int_\Gamma\frac{dw}{2\pi i}\frac{f(w)}{w-u}$

$$\chi(u)\equiv\left(x(u)-\frac{1}{x(u)}\right)\xi(u),\ \ F(u)\equiv\left(x(u)-\frac{1}{x(u)}\right)f(u)\ \ \to\ \ \chi(u+i0)+\chi(u-i0)=F(u)\ \ \to\ \ \chi(u)=K_u\star F\ \ \equiv\ \int_{-2g+i0}^{2g+i0}\frac{dw}{2\pi i}\frac{x(u)-\frac{1}{x(u)}}{x(w)-\frac{1}{x(w)}}\frac{1}{w-u}F(w)$$

$$\chi(u,v)=\frac{1}{i}K_v\star K_u\star\frac{\Gamma(1+iu-iv)}{\Gamma(1-iu+iv)}$$

Zhukovsky map

$$z=x(w),\quad z+\frac{1}{z}=\frac{w}{g},\quad x=x(u),\quad x+\frac{1}{x}=\frac{u}{g}$$

$$\begin{aligned}K_u\star F\ &\equiv\ \int_{-2g+i0}^{2g+i0}\frac{dw}{2\pi i}\frac{x(u)-\frac{1}{x(u)}}{x(w)-\frac{1}{x(w)}}\frac{1}{w-u}F(w)\\&=\oint_{|z|=1}\frac{dz}{2\pi i}\frac{1}{x-z}F\left(g(z+1/z)\right)-\frac{1}{g}\int_{-2g+i0}^{2g+i0}\frac{dw}{2\pi i}\frac{1}{x(w)-\frac{1}{x(u)}}F(w)\end{aligned}$$

$$K_v\star K_u\star F=\oint_{|z'|=1}\frac{dz'}{2\pi i}\frac{1}{y-z'}\oint_{|z|=1}\frac{dz}{2\pi i}\frac{1}{x-z}F\left(g(z+1/z),g(z'+1/z')\right)+(\text{symmetric in }u\leftrightarrow v,\ \ x\leftrightarrow y)$$

- After anti-symmetrization of χ : integral form of the dressing factor

$$\chi(x,y) = -i \oint_{|z|=1} \frac{dz}{2\pi i} \oint_{|z'|=1} \frac{dz'}{2\pi i} \frac{1}{x-z} \frac{1}{y-z'} \frac{\ln\Gamma\left[1+ig\left(z_1+\frac{1}{z_1}-z_2-\frac{1}{z_2}\right)\right]}{\ln\Gamma\left[1-ig\left(z_1+\frac{1}{z_1}-z_2-\frac{1}{z_2}\right)\right]}$$

- Weak coupling expansion $z = e^{i\phi},\ z' = e^{i\phi'}$

$$\begin{aligned} \chi(x,y) &= -\sum_{r,s=1}^{\infty} \frac{c_{r,s}(g)}{x^r y^s} \\ c_{r,s}(g) &= i \int_0^{2\pi} \frac{d\phi}{2\pi} \int_0^{2\pi} \frac{d\phi'}{2\pi} e^{ir\phi+is\phi'} \frac{\ln\Gamma\left[1+2ig(\cos\phi-\cos\phi')\right]}{\ln\Gamma\left[1-2ig(\cos\phi-\cos\phi')\right]} \\ &= 2\sin\left(\frac{\pi}{2}(r-s)\right) \int_0^{\infty} dt \frac{J_r(2gt)J_s(2gt)}{t(e^t-1)} \end{aligned}$$

$$c_{r,s}(g) = \sum_{n=1}^{\infty} g^{r+s+2n} \cdot 2(-1)^n \sin\left(\frac{\pi}{2}(r-s)\right) \frac{(2n+r+s-1)!(2n+r+s)!}{n!(n+r)!(n+s)!(n+r+s)!} \zeta(2n+r+s)$$

$$\begin{aligned} \sigma^2(u,v) &= \exp\left\{2i\left[\chi(x^+,y^-)+\chi(x^-,y^+)-\chi(x^+,y^+)-\chi(x^-,y^-)\right]\right\} \\ &= 1+256\zeta(3)g^6 \frac{(u-v)(4uv-1)}{(1+4u^2)^2(1+4v^2)^2} + \mathcal{O}(g^8) \end{aligned}$$

- Strong coupling expansion

$$c_{r,s}(g) = \sum_{n=1}^{\infty} g^{1-n} \cdot \frac{\zeta(n)((-1)^{r+s}-1)\Gamma(\frac{1}{2}(n-r+s-1))\Gamma(\frac{1}{2}(n+r+s-3))}{2(-2\pi)^n\Gamma(n-1)\Gamma(\frac{1}{2}(-n-r+s+3))\Gamma(\frac{1}{2}(-n+r+s+1))}$$
$$= g\frac{\delta_{s,r-1}-\delta_{s,r+1}}{rs} + \frac{(-1)^{r+s}-1}{\pi}\frac{1}{r^2-s^2} + \mathcal{O}(g^{-1})$$

$$\chi^{(0)}(x,y) = \left(x+\frac{1}{x}-y-\frac{1}{y}\right)\ln\left(1-\frac{1}{xy}\right)-\frac{1}{x}+\frac{1}{y}$$

$$\sigma(u,v) \approx \frac{1-\frac{1}{x^-y^+}}{1-\frac{1}{x^+y^-}}\left(\frac{1-\frac{1}{x^-y^-}}{1-\frac{1}{x^-y^+}}\,\frac{1-\frac{1}{x^+y^+}}{1-\frac{1}{x^+y^-}}\right)^{i(v-u)}$$

Arutyunov, Frolov, Staudacher (2004)

- Compare with the classical string result from the sine-Gordon scattering

$$S(p_1,p_2) = e^{i\delta(p_1,p_2)} = \left(\frac{\sin^2\frac{p_1+p_2}{4}}{\sin^2\frac{p_1-p_2}{4}}\right)^{4ig\left(\cos\frac{p_1}{2}-\cos\frac{p_2}{2}\right)}$$

- From exact S-matrix

$$S(p_1,p_2) = A(p_1,p_2)^2 = S_0^2\left(\frac{x_2^- - x_1^+}{x_2^+ - x_1^-}\right)^2 \approx \left(\frac{1-\frac{1}{x_1^-x_2^-}}{1-\frac{1}{x_1^-x_2^+}}\,\frac{1-\frac{1}{x_1^+x_2^+}}{1-\frac{1}{x_1^+x_2^-}}\right)^{2i(u_1-u_2)}$$

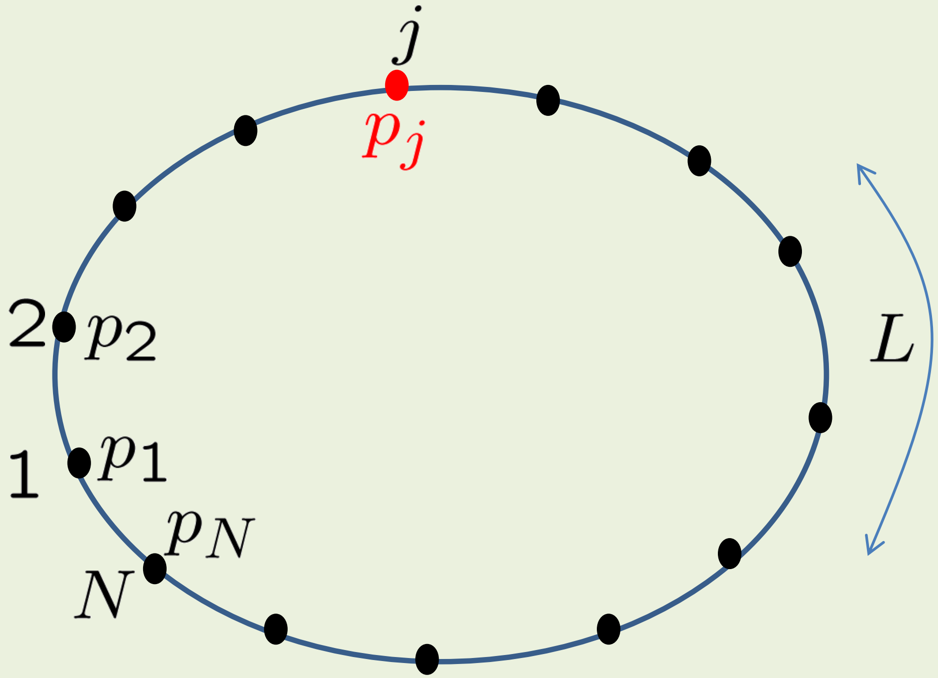
$$S_0(p_1,p_2)^2 \equiv \frac{x_1^- - x_2^+}{x_1^+ - x_2^-}\frac{1-\frac{1}{x_1^+x_2^-}}{1-\frac{1}{x_1^-x_2^+}}\sigma(p_1,p_2)^2$$

$$x_j^\pm \approx e^{\pm i\frac{p_j}{2}}, \qquad u_j \approx 2g\cos\frac{p_j}{2}$$

Asymptotic Bethe ansatz equations

Periodic BC

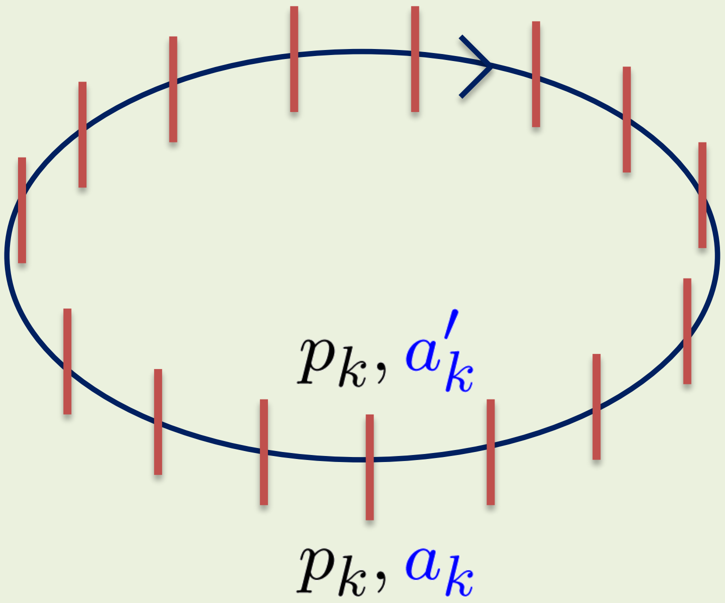
- Going around the world

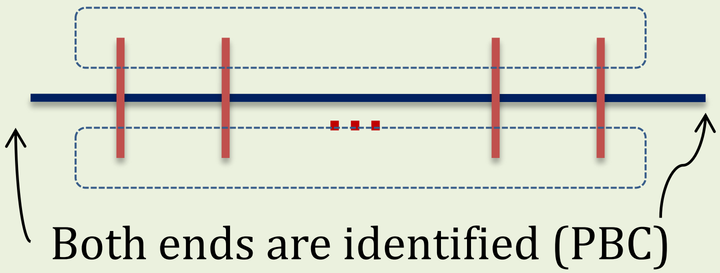


- At each crossing, S-matrix

$$e^{ip_jL}\prod_{k\neq j,1}^{N} S(p_j,p_k)=1$$

- When the scattering is non-diagonal



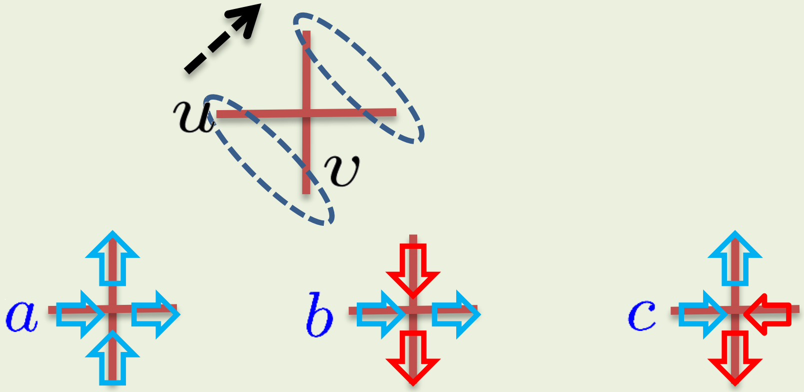


$$e^{ip_jL} S^{a'_1 b_2}_{a_1 b_1}(p_1, p_j) S^{a'_2 b_3}_{a_2 b_2}(p_2, p_j) \cdots S^{a'_N b_1}_{a_N b_N}(p_N, p_j) = 1$$

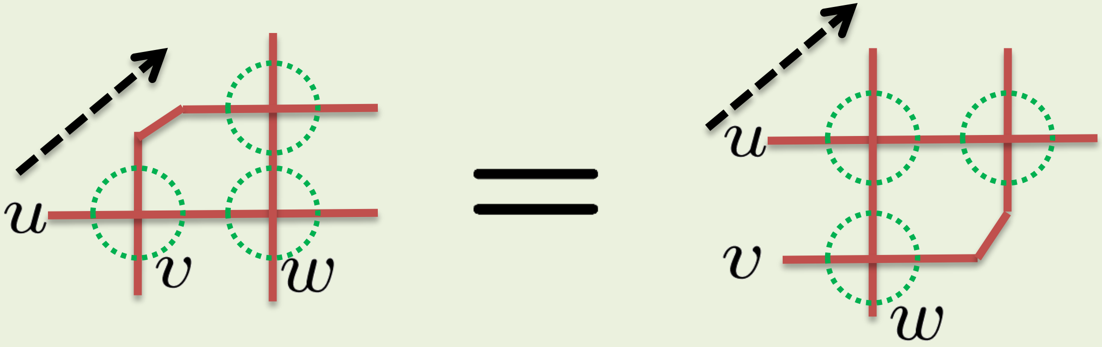
- Transfer matrix $e^{ip_jL} \cdot T(p_j|\boxed{p_1, \cdots, p_N})^{\vec{a}'}_{\vec{a}} = 1$
- Need to diagonalize transfer matrix [Lecture by Nepomechie]
  - Main difference is “inhomogeneity”

su(2)-invariant S-matrix

$$S(u-v) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}$$



- S should satisfy YBE

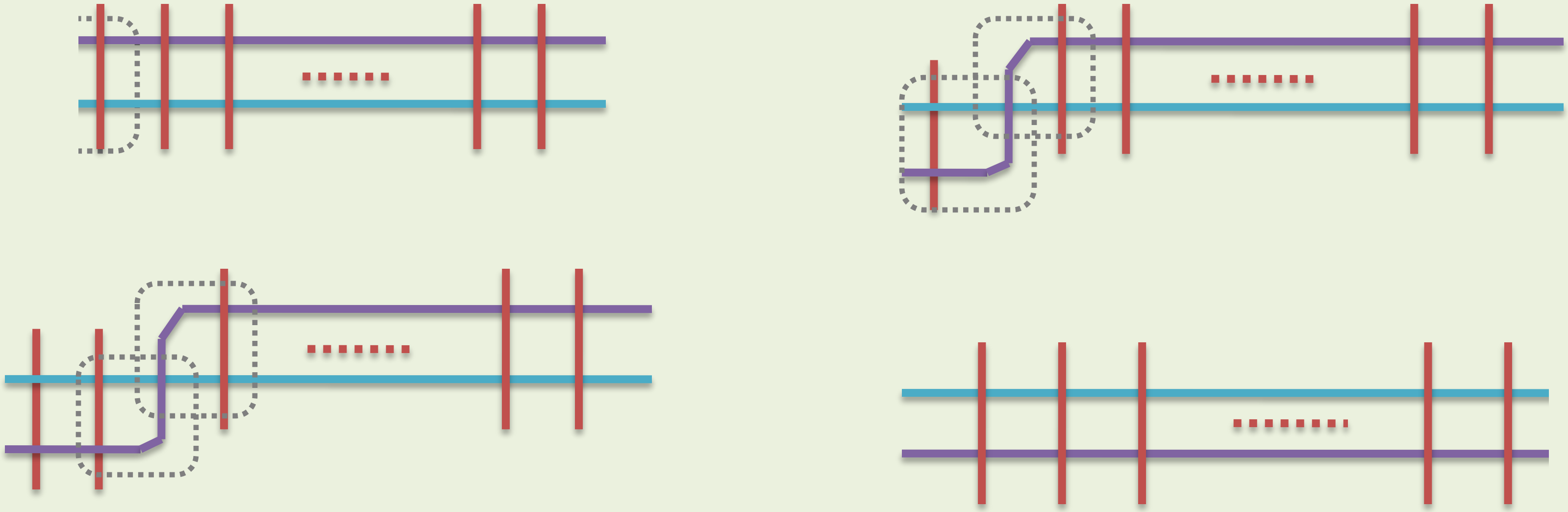


A Solution of YBE

- su(2)-invariant

$$a(u) = u + i,\ \ b(u) = u,\ \ c(u) = i$$

- Transfer matrices commute → Integrable



Algebraic Bethe ansatz

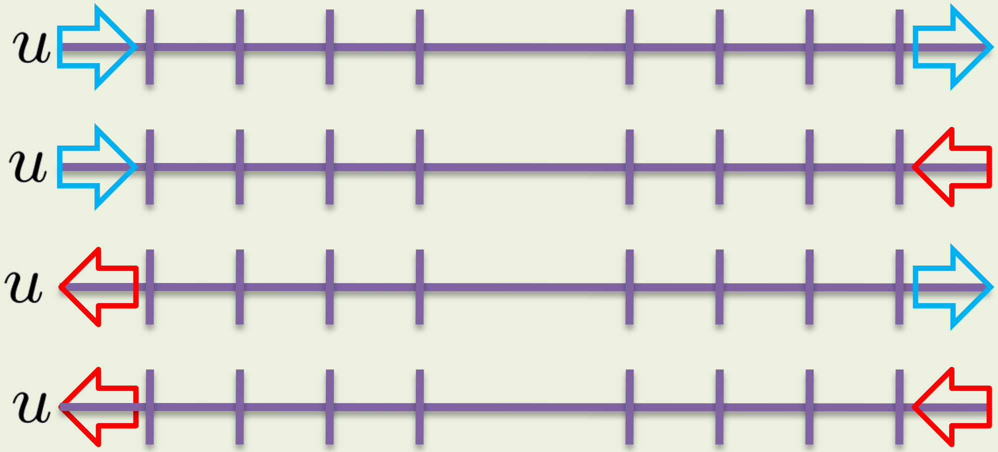
- Monodromy matrix :

$\mathcal{A}(u)$:

$\mathcal{B}(u)$:

$\mathcal{C}(u)$:

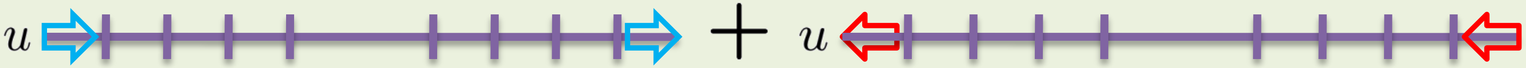
$\mathcal{D}(u)$:



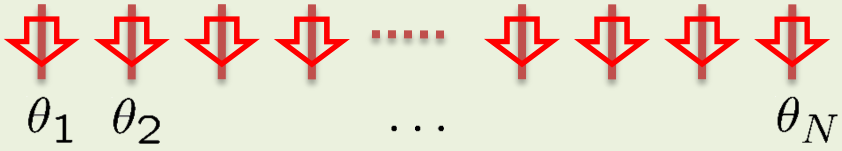
- Transfer matrix for the PBC

$$T(u) = \mathrm{Tr}\left[\begin{pmatrix} \mathcal{A}(u) & \mathcal{B}(u) \\ \mathcal{C}(u) & \mathcal{D}(u) \end{pmatrix}\right] = \mathcal{A}(u) + \mathcal{D}(u)$$

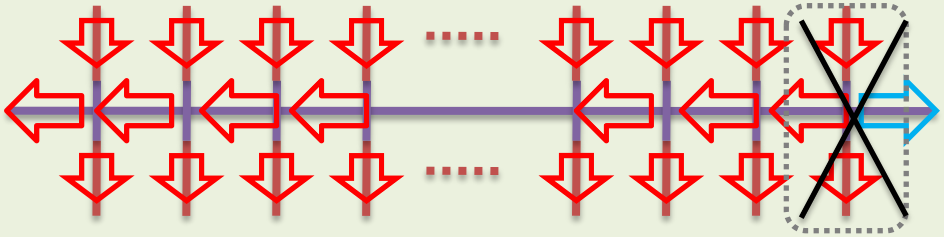
$T(u) =$



- Ferromagnetic vacuum state $|\Downarrow\Downarrow\Downarrow\cdots\Downarrow\rangle$



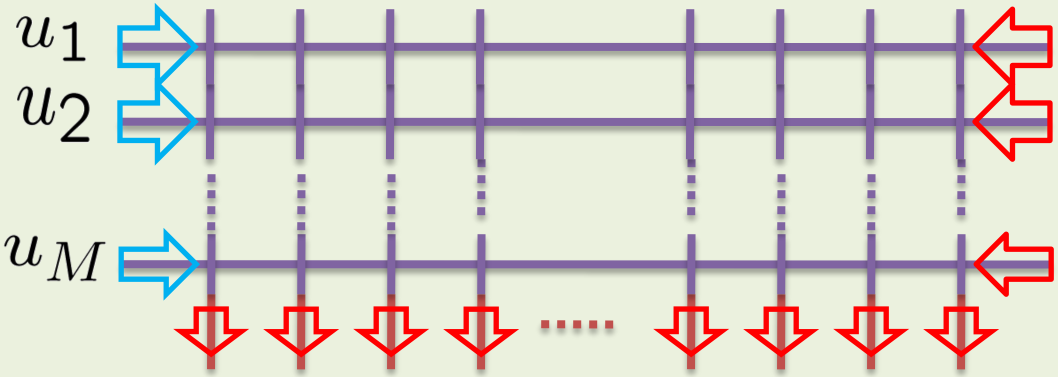
- Annihilation operator $\mathcal{C}(u)|\Downarrow\Downarrow\Downarrow\cdots\Downarrow\rangle = 0$



- Creation operator $\mathcal{B}(u)$

- Construct a general state

$$|\Psi(u_1,\cdots,u_M)\rangle = \mathcal{B}(u_1)\mathcal{B}(u_2)\cdots\mathcal{B}(u_M)|\overset{\theta_1}{\color{red}\Downarrow}{\color{red}\Downarrow\Downarrow}\overset{\cdots}{\cdots}\overset{\theta_N}{\color{red}\Downarrow}\rangle$$

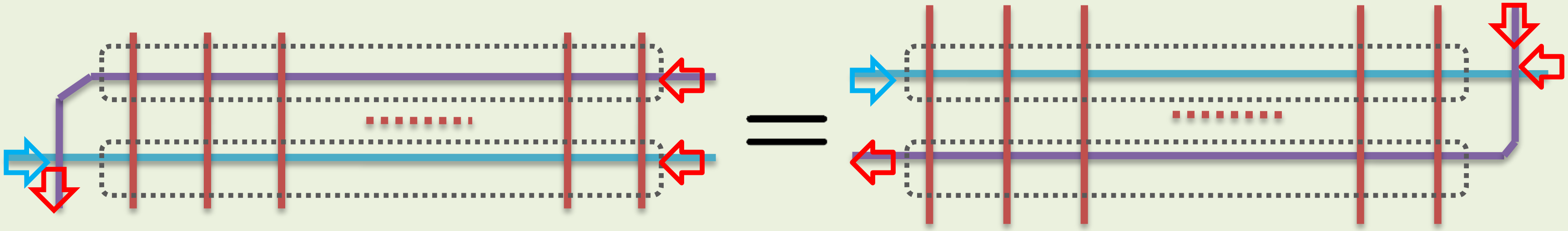


- Act the transfer matrix

$$\color{blue}T\color{black}(u)|\Psi(u_1,\cdots,u_M)\rangle = [\mathcal{A}(u)+\mathcal{D}(u)]\mathcal{B}(u_1)\mathcal{B}(u_2)\cdots\mathcal{B}(u_M)|\overset{\theta_1}{\color{red}\Downarrow}{\color{red}\Downarrow\Downarrow}\overset{\cdots}{\cdots}\overset{\theta_N}{\color{red}\Downarrow}\rangle$$

$\color{blue}T\color{black}(u) = u$  $+ \; u$ 

- We have seen that



$$b(u-v)\mathcal{D}(v)\mathcal{B}(u)+c(u-v)\mathcal{B}(v)\mathcal{D}(u) = a(u-v)\mathcal{B}(u)\mathcal{D}(v)$$

$$b(u-v)\mathcal{A}(v)\mathcal{B}(u)+c(u-v)\mathcal{B}(v)\mathcal{A}(u) = a(u-v)\mathcal{B}(u)\mathcal{A}(v)$$

- Act *A* & *D* on the state |Ψ> using CR

$$\mathcal{A}(u)\mathcal{B}(u_1)\mathcal{B}(u_2)\cdots\mathcal{B}(u_M)|\overset{\theta_1}{\Downarrow}\Downarrow\Downarrow\overset{\cdots}{\cdots}\overset{\theta_N}{\Downarrow}\rangle$$

$$\mathcal{A}(u)\mathcal{B}(u_j) = \boxed{\frac{a(u_j-u)}{b(u_j-u)}\mathcal{B}(u_j)\mathcal{A}(u)} - \frac{c(u_j-u)}{b(u_j-u)}\mathcal{B}(u)\mathcal{A}(u_j)$$

$$\mathcal{D}(u)\mathcal{B}(u_1)\mathcal{B}(u_2)\cdots\mathcal{B}(u_M)|\overset{\theta_1}{\Downarrow}\Downarrow\Downarrow\overset{\cdots}{\cdots}\overset{\theta_N}{\Downarrow}\rangle$$

$$\mathcal{D}(u)\mathcal{B}(u_j) = \boxed{\frac{a(u-u_j)}{b(u-u_j)}\mathcal{B}(u_j)\mathcal{D}(u)} - \frac{c(u-u_j)}{b(u-u_j)}\mathcal{B}(u)\mathcal{D}(u_j)$$

- Only “wanted” terms contribute

- Eigenvalues

$$\Lambda(u) = \prod_{n=1}^{N} a(u-\theta_n) \prod_{j=1}^{M} \frac{a(u_j - u)}{b(u_j - u)} + \prod_{n=1}^{N} b(u-\theta_n) \prod_{j=1}^{M} \frac{a(u - u_j)}{b(u - u_j)}$$

- BAE: $\Lambda(u_k) = \text{finite}, \qquad b(u_k - u_k) = 0$

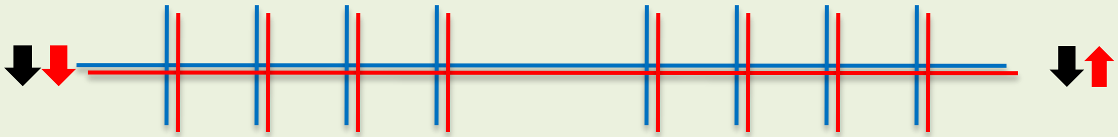
$$\prod_{n=1}^{N} \frac{a(u_k - \theta_n)}{b(u_k - \theta_n)} = - \prod_{j=1}^{M} \frac{a(u_k - u_j)}{a(u_j - u_k)} = \prod_{j\neq k, j=1}^{M} \frac{u_k - u_j + i}{u_k - u_j - i}$$

- Bethe-Yang equation:

$$e^{ip(\theta_j)L}\Lambda(\theta_j) = e^{ip(\theta_j)L} \prod_{n=1}^{N} a(\theta_j - \theta_n) \prod_{k=1}^{M} \frac{a(u_k - \theta_j)}{b(u_k - \theta_j)} = 1$$

Asymptotic BAE for AdS/CFT

- S-matrix is related to R-matrix of 1d Hubbard model **Beisert**
  - Both have $su(2)\times su(2)$ $\boxslash=(\phi_a|\psi_\alpha)=(\phi_1,\phi_2|\psi_3,\psi_4)=(\Uparrow,\Downarrow)\otimes(\Uparrow,\Downarrow)$
- Algebraic Bethe ansatz is applicable **Martins, Ramos**
- Monodromy matrix : 4 x 4



$$\mathcal{T}=\begin{pmatrix} \mathcal{B} & \mathcal{B}_1 & \mathcal{B}_2 & \mathcal{F} \\ \mathcal{C}_1 & \mathcal{A}_{11} & \mathcal{A}_{12} & \mathcal{B}_1^* \\ \mathcal{C}_2 & \mathcal{A}_{21} & \mathcal{A}_{22} & \mathcal{B}_2^* \\ \mathcal{C} & \mathcal{C}_1^* & \mathcal{C}_2^* & \mathcal{D} \end{pmatrix}$$

- Transfer matrix $\mathbf{T}=\mathrm{sTr}\,[\mathcal{T}]=\mathcal{B}+\mathcal{D}-(\mathcal{A}_{11}+\mathcal{A}_{22})$
- Vacuum: $|0\rangle=|(\Uparrow\Uparrow)(\Uparrow\Uparrow)\cdots(\Uparrow\Uparrow)\rangle$

- One-particle excited states $|\Phi(u_1)\rangle = [\mathcal{B}_a(u_1)F^a]\,|0\rangle$

Constant vector

$$\begin{aligned} \mathcal{B}(u)|\Phi(u_1)\rangle &= \alpha_1\ \omega_1(u)^L\ |\Phi(u_1)\rangle + \dots \\ \mathcal{D}(u)|\Phi(u_1)\rangle &= \alpha_2\ \omega_2(u)^L\ |\Phi(u_1)\rangle + \dots \\ \mathcal{A}_{aa}(u)|\Phi(u_1)\rangle &= \alpha_3\ \mathbf{r}^{ac}_{ba}\ \mathcal{B}_c(u_1)\ F^b\ \omega_3(u)^L\ |0\rangle + \dots \\ &= \alpha_3\ \Lambda^{(1)}\ \omega_3(u)^L|\Phi(u_1)\rangle + \dots \end{aligned}$$

su(2) R-matrix

$$\mathbf{r}^{ac}_{ba}\ F^b = \Lambda^{(1)}F^c$$

- Multi-particle states $|\Phi\rangle = [\mathcal{B}_a(u_1)F^a]\cdots[\mathcal{B}_a(u_M)F^a]\,|0\rangle$

$$\begin{aligned} \mathcal{B}(u)|\Phi\rangle &= \left[\prod_{j=1}^{M}\alpha_1(u,u_j)\right]\ \omega_1(u)^L\ |\Phi\rangle + \dots \\ \mathcal{D}(u)|\Phi\rangle &= \left[\prod_{j=1}^{M}\alpha_2(u,u_j)\right]\ \omega_2(u)^L\ |\Phi\rangle + \dots \\ \mathcal{A}_{aa}(u)|\Phi\rangle &= \left[\prod_{j=1}^{M}\alpha_3(u,u_j)\mathbf{T}^{(1)}(u,u_j)\right]\ \omega_3(u)^L\ |\Phi\rangle + \dots \end{aligned}$$

su(2) transfer matrix

- Eigenvalues

one BAE &

$$\Lambda(u) = \left[\prod_{j=1}^{M}\alpha_1(u,u_j)\right]\ \omega_1(u)^L + \left[\prod_{j=1}^{M}\alpha_2(u,u_j)\right]\ \omega_2(u)^L - \left[\prod_{j=1}^{M}\alpha_3(u,u_j)\Lambda^{(1)}(u,u_j;\lambda_l)\right]\ \omega_3(u)^L$$

- Inhomogeneity since vacuum consists of particles with momenta

$$\Lambda(u) = \underbrace{\left[\prod_{j=1}^{M} \alpha_1(u,u_j)\right] \left[\prod_{k=1}^{L} \omega_1(u,\theta_k)\right] + \left[\prod_{j=1}^{M} \alpha_2(u,u_j)\right] \left[\prod_{k=1}^{L} \omega_2(u,\theta_k)\right] + \left[\prod_{j=1}^{M} \alpha_3(u,u_j)\Lambda^{(1)}(u,u_j;\lambda_l)\right]}_{} \left[\prod_{k=1}^{L} \omega_3(u,\theta_k)\right]$$

- Another BAE from $\Lambda(u_n) = \text{finite}$ with $\alpha_1(u_j,u_j) = \alpha_3(u_j,u_j) = \infty$

$$\prod_{k=1}^{L} \frac{\omega_1(u_j,\theta_k)}{\omega_3(u_j,\theta_k)} = \Lambda^{(1)}(\{u_k\};\{\lambda_l\})$$

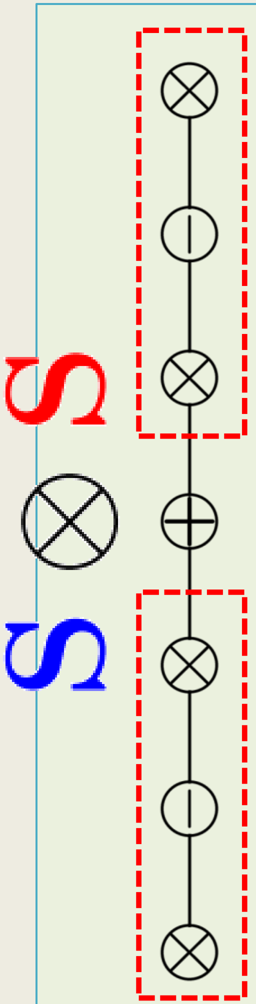
- Bethe-Yang equation: $\omega_2(\theta_k,\theta_k) = \omega_3(\theta_k,\theta_k) = 0$

$$e^{ip(\theta_j)L}\Lambda(\theta_j) = e^{ip(\theta_j)L} \left[\prod_{k=1}^{M} \alpha_1(\theta_j,u_k)\right] \left[\prod_{k=1}^{L} \omega_1(\theta_j,\theta_k)\right] = 1$$

- Considering the two S-matrices: two wings of BAE and Bethe-Yang equation

$$e^{ip(\theta_j)L}{\color{blue}\Lambda}(\theta_j,\{{\color{blue}u_k}\}){\color{red}\Lambda}(\theta_j,\{{\color{red}u_k}\}) = 1$$

Asymptotic Bethe-Yang equation



$$
\begin{aligned}
1 &= \prod_{k=1}^{K_2} \frac{u_{1j}-u_{2k}+\frac{i}{2}}{u_{1j}-u_{2k}-\frac{i}{2}} \prod_{k=1}^{K_4} \frac{1-1/x_{1j}x^+_{4k}}{1-1/x_{1j}x^-_{4k}} \\
1 &= \prod_{k=1}^{K_2} \frac{u_{2j}-u_{2k}-i}{u_{2j}-u_{2k}+i} \prod_{k=1}^{K_3} \frac{u_{2j}-u_{3k}+\frac{i}{2}}{u_{2j}-u_{3k}-\frac{i}{2}} \prod_{k=1}^{K_1} \frac{u_{2j}-u_{1k}+\frac{i}{2}}{u_{2j}-u_{1k}-\frac{i}{2}} \\
1 &= \prod_{k=1}^{K_2} \frac{u_{3j}-u_{2k}+\frac{i}{2}}{u_{3j}-u_{2k}-\frac{i}{2}} \prod_{k=1}^{K_4} \frac{x_{3j}-x^+_{4k}}{x_{3j}-x^-_{4k}} \\
\left(\frac{x^+_{4j}}{x^-_{4j}}\right)^L &= \prod_{k=1}^{K_4} \sigma^2(x_{4j},x_{4k}) \frac{u_{4j}-u_{4k}+i}{u_{4j}-u_{4k}-i} \\
&\times \prod_{k=1}^{K_1} \frac{1-1/x^-_{4j}x_{1k}}{1-1/x^+_{4j}x_{1k}} \prod_{k=1}^{K_3} \frac{x^-_{4j}-x_{3k}}{x^+_{4j}-x_{3k}} \prod_{k=1}^{K_5} \frac{x^-_{4j}-x_{5k}}{x^+_{4j}-x_{5k}} \prod_{k=1}^{K_7} \frac{1-1/x^-_{4j}x_{7k}}{1-1/x^+_{4j}x_{7k}} \\
1 &= \prod_{k=1}^{K_6} \frac{u_{5j}-u_{6k}+\frac{i}{2}}{u_{5j}-u_{6k}-\frac{i}{2}} \prod_{k=1}^{K_4} \frac{x_{5j}-x^+_{4k}}{x_{5j}-x^-_{4k}} \\
1 &= \prod_{k=1}^{K_6} \frac{u_{6j}-u_{6k}-i}{u_{6j}-u_{6k}+i} \prod_{k=1}^{K_5} \frac{u_{6j}-u_{5k}+\frac{i}{2}}{u_{6j}-u_{5k}-\frac{i}{2}} \prod_{k=1}^{K_7} \frac{u_{6j}-u_{7k}+\frac{i}{2}}{u_{6j}-u_{7k}-\frac{i}{2}} \\
1 &= \prod_{k=1}^{K_6} \frac{u_{7j}-u_{6k}+\frac{i}{2}}{u_{7j}-u_{6k}-\frac{i}{2}} \prod_{k=1}^{K_4} \frac{1-1/x_{7j}x^+_{4k}}{1-1/x_{7j}x^-_{4k}}
\end{aligned}
$$

Simpler form of BAE

- Dynamic transformation **Beisert, Roiban; Hentschel, Plefka, Sundin**

$$K_{1,7}\to K_{1,7}-1,\quad K_{3,5}\to K_{3,5}+1,\quad L\to L-1$$

$$L' = L - K_1 - K_7$$

$$\begin{aligned}
1 &= \prod_{k=1}^{K_2}\frac{u_{2j}-u_{2k}-i}{u_{2j}-u_{2k}+i}\prod_{k=1}^{K_3+K_1}\frac{u_{2j}-u_{3k}+\frac{i}{2}}{u_{2j}-u_{3k}-\frac{i}{2}}\\
1 &= \prod_{k=1}^{K_2}\frac{u_{3j}-u_{2k}+\frac{i}{2}}{u_{3j}-u_{2k}-\frac{i}{2}}\prod_{k=1}^{K_4}\frac{x_{3j}-x^+_{4k}}{x_{3j}-x^-_{4k}}\\
\left(\frac{x^+_{4j}}{x^-_{4j}}\right)^{L'} &= \prod_{k=1}^{K_4}\sigma^2(x_{4j},x_{4k})\frac{u_{4j}-u_{4k}+i}{u_{4j}-u_{4k}-i}\prod_{k=1}^{K_3+K_1}\frac{x^-_{4j}-x_{3k}}{x^+_{4j}-x_{3k}}\prod_{k=1}^{K_5+K_7}\frac{x^-_{4j}-x_{5k}}{x^+_{4j}-x_{5k}}\\
1 &= \prod_{k=1}^{K_6}\frac{u_{5j}-u_{6k}+\frac{i}{2}}{u_{5j}-u_{6k}-\frac{i}{2}}\prod_{k=1}^{K_4}\frac{x_{5j}-x^+_{4k}}{x_{5j}-x^-_{4k}}\\
1 &= \prod_{k=1}^{K_6}\frac{u_{6j}-u_{6k}-i}{u_{6j}-u_{6k}+i}\prod_{k=1}^{K_5+K_7}\frac{u_{6j}-u_{5k}+\frac{i}{2}}{u_{6j}-u_{5k}-\frac{i}{2}}
\end{aligned}$$

Lecture 3. Finite-size effects

Plan

1. Wrapping effect

2. Luscher corrections

3. Thermodynamic Bethe ansatz method

4. Y-systems

Four-loop su(2) Konishi

- su(2) Konishi $\mathrm{Tr}\,[ZZXX]\,,\qquad \mathrm{Tr}\,[ZXZX]$
- BAE : $p_1 = -p_2 = p, \qquad \sigma = e^{2i\theta(p,-p)}$

$$e^{i4p} = e^{-i72\sqrt{3}\zeta(3)g^6}\cdot\frac{2u+i}{2u-i}, \qquad u = \frac{1}{2}\cot\frac{p}{2}\sqrt{1+16g^2\sin^2\frac{p}{2}}$$

$$\uparrow$$

$$\sigma^2(u,v) \approx 1 + 256\zeta(3)g^6\frac{(u-v)(4uv-1)}{(1+4u^2)^2(1+4v^2)^2}, \qquad u = -v = \frac{1}{2\sqrt{3}}$$

- Perturbative solutions $p = \frac{2\pi}{3} - \sqrt{3}g^2 + \frac{9\sqrt{3}}{2}g^4 - \frac{72\sqrt{3}+72\sqrt{3}\zeta(3)}{3}g^6 + \ldots$
- BAE result:

$$\Delta_{\mathrm{BAE}} = 4 + 12g^2 - 48g^4 + 336g^6 - \boxed{(2820 + 288\zeta(3))}g^8 + \mathcal{O}(g^{10})$$

- Perturbative SYM calculation **Fiamberti, Santambrogio, Sieg, Zanon (2008)**

$$\Delta_{\mathrm{Pert.}} = 4 + 12g^2 - 48g^4 + 336g^6 - \boxed{(2496 - 576\zeta(3) + 1440\zeta(5))}g^8 + \mathcal{O}(g^{10})$$

- BAE is wrong at the 4-loop level

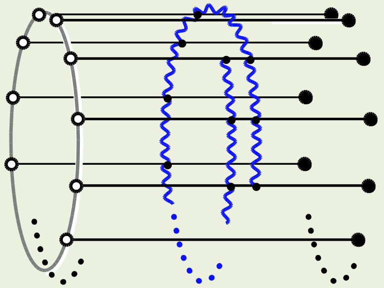
$$\delta\Delta = \Delta_{\mathrm{Pert.}} - \Delta_{\mathrm{BAE}} = (324 + 864\zeta(3) - 1440\zeta(5))g^8 + \mathcal{O}(g^{10})$$

- WHY? Finite-size effect !

Wrapping problem

- High-order Feynman diagrams connect operators farther away
- When the length of a composite operator is shorter than the order of the perturbative expansion: unphysical(“wrapping”) interactions appear

  → BAE is valid only when the length is infinite
- The length of spin-chain is another important parameter



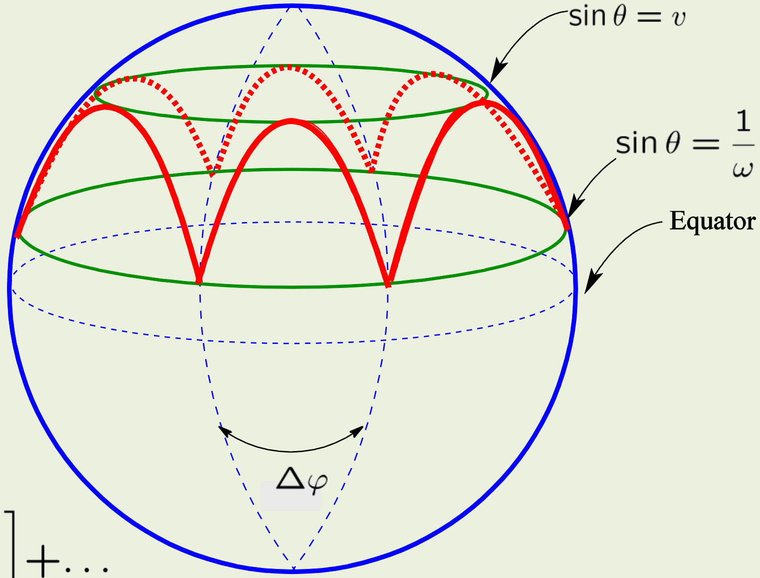
Giant magnon in R x S^2

- sine-Gordon model with a finite size

Arutyunov-Frolov-Zamaklar

$$\cos\theta \;=\; \sqrt{1-v^2}\,\mathrm{dn}\left(\frac{1}{\sqrt{\eta}}\frac{\sigma - v\omega\tau}{\sqrt{1-v^2\omega^2}},\eta\right), \qquad \eta = \frac{1-\omega^2 v^2}{\omega^2(1-v^2)}$$

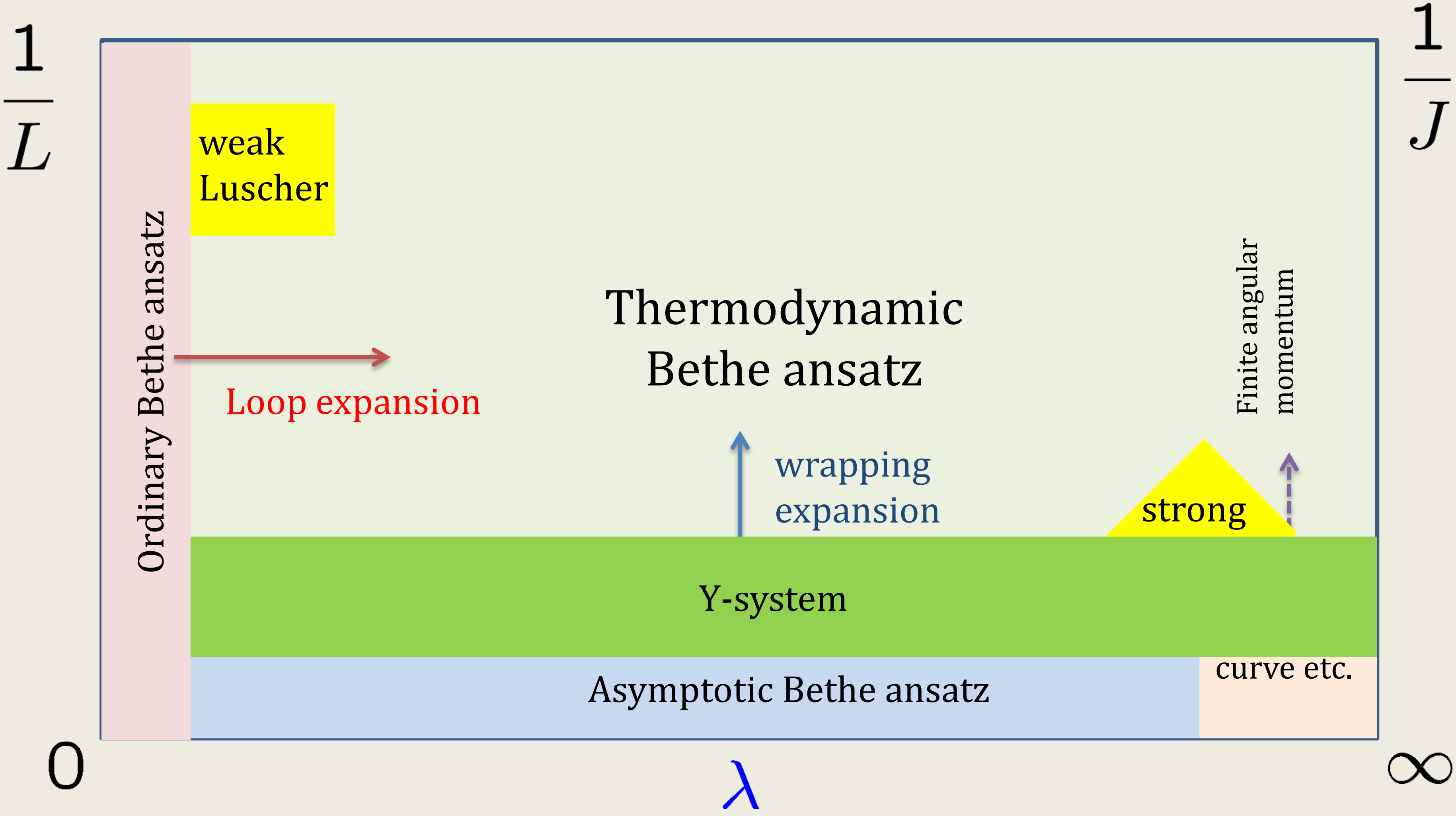
$$\approx \frac{\sin\frac{p}{2}}{\cosh\xi}, \qquad \left[\eta\to 1, \quad v \to \cos\frac{p}{2}, \quad \omega \to 1\right]$$



- Correction to E-J relation

$$E-J \approx 4g\sin\frac{p}{2} \underbrace{-16g\sin^3\frac{p}{2}\ \exp\left[-\left(\frac{J}{2g\sin\frac{p}{2}}+2\right)\right]}_{\text{Finite-size effect}} +\ldots$$

Phase diagram of integrable methods

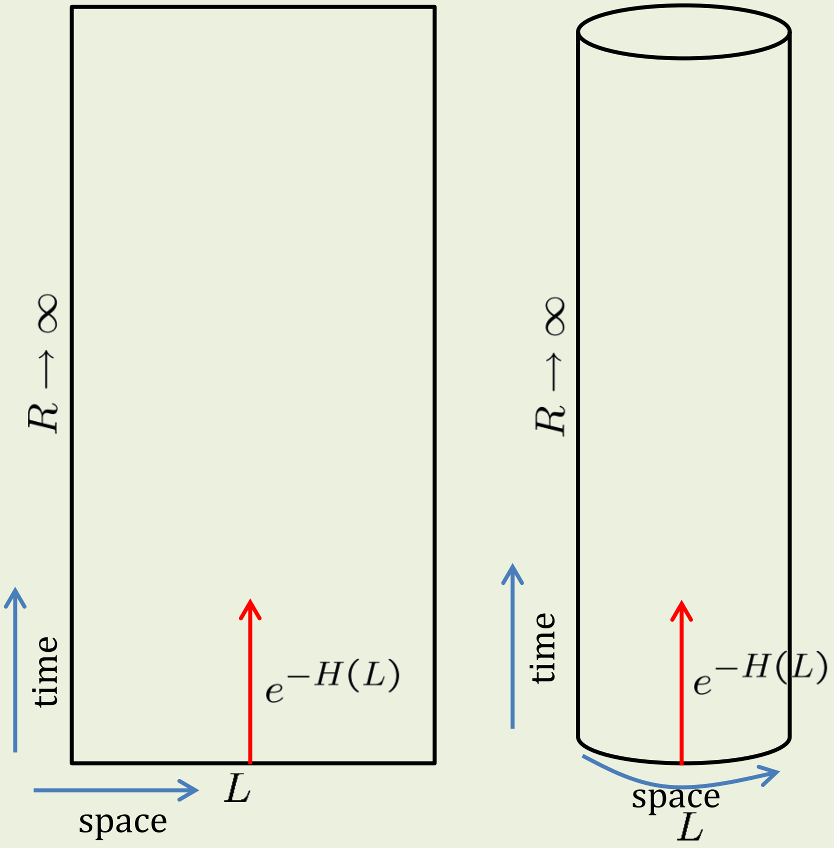


Thermodynamic Bethe ansatz

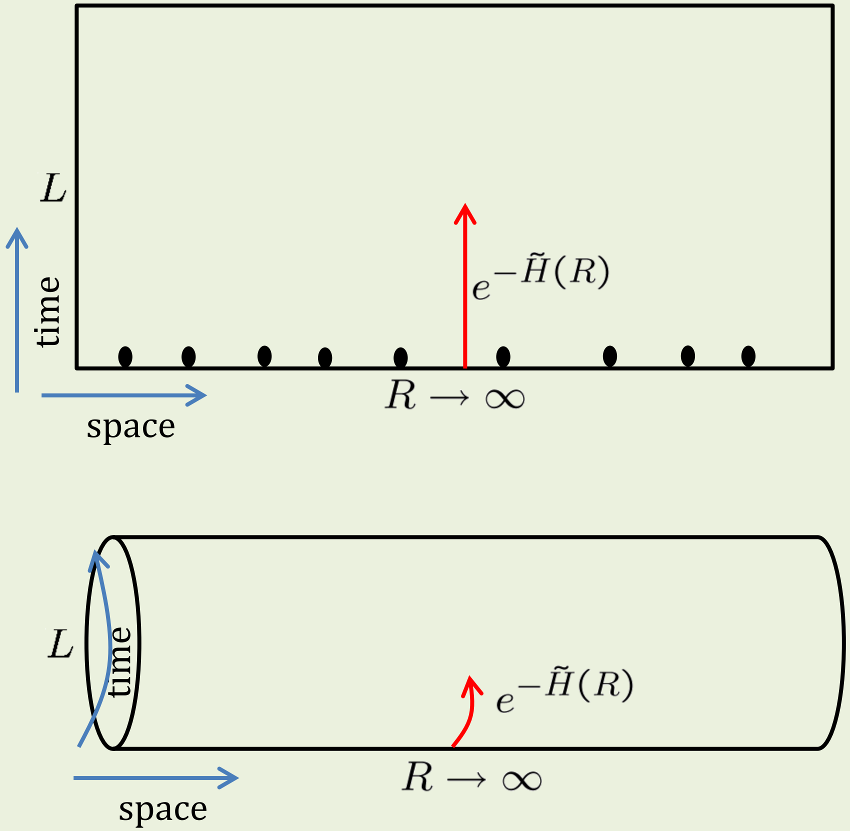
- From S-matrix to the finite-size effect
- Al. B. Zamolodchikov (1990)

2d Euclidean geometry with PBC

Physical space



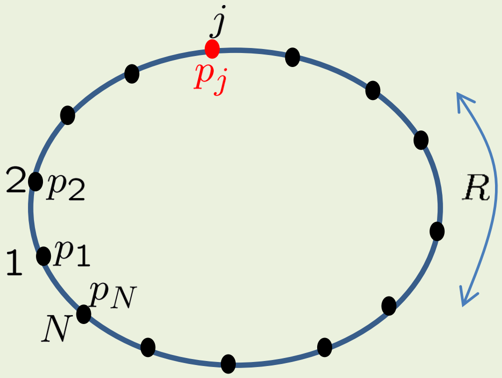
Mirror space



Channel duality

- Mirror channel
  - An elementary excitation with a dispersion relation $(\tilde{e}(u), \tilde{p}(u))$
  - *S*-matrix and scattering are valid only when *R*→∞
  - *N*-particles in a box of length *R*
  - Bethe-Yang equation $e^{i\tilde{p}(u_j)R} \prod_{k\neq j,1}^{N} S(u_j, u_k) = 1$
  - Partition function

$$\tilde{Z}(R,L) = \mathrm{Tr}\left[e^{-L\tilde{H}(R)}\right]$$



- Physical channel
  - Dispersion relation $(e,p) = (-i\tilde{p}, -i\tilde{e})$
  - Partition function $Z(L,R) = \mathrm{Tr}\left[e^{-RH(L)}\right] \approx e^{-RE_0(L)}$ as $R \to \infty$

$$\tilde{Z}(R,L) = Z(L,R) \quad \rightarrow \quad E_0(L) = -\frac{1}{R}\ln \tilde{Z}(R,L) = \frac{L}{R}\tilde{\mathcal{F}}(L)$$

Free energy with temperature

$$T = \frac{1}{L}$$

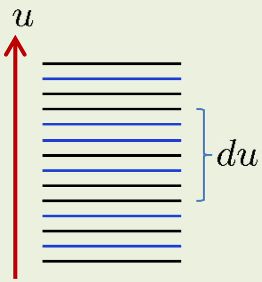
  - Computing free energy in the mirror space

$$\tilde{\mathcal{F}}(L) = \tilde{E} - T\mathcal{S}$$

- Mirror free energy with $N,\ R \to \infty$
  - Log of Bethe-Yang equation :

$$\tilde{p}(u_j) - \frac{i}{R}\sum_{k\neq j,1}^{N} \ln S(u_j,u_k) = 2\pi\frac{n_j}{R} \;\rightarrow\; \tilde{p}(u_j) + \int u'\rho(u')\frac{1}{i}\ln S(u_j,u') = 2\pi\frac{\overset{\overset{\text{any integer}}{\downarrow}}{n_j}}{R}$$

$$\rightarrow \boxed{\frac{d\tilde{p}}{du} + \int du'\rho(u')\frac{1}{i}\frac{\partial}{\partial u}\ln S(u,u') = 2\pi[\rho_h(u)+\rho(u)]}$$



  - Energy $\tilde{E} = \sum_{j=1}^{N}\tilde{e}(u_j) = R\int du\,\rho(u)\,\tilde{e}(u), \quad \rho(u) = \frac{1}{R}\frac{dn}{du}, \quad \rho_h(u) = \frac{1}{R}\frac{dn}{du}$
  - *n* = # of particles,
  - *dn* = # of particles with u-values between *u* and *u+du*
  - *n* = # of unoccupied ('holes') states
  - Entropy : log of # of cases $\mathcal{S} = R\int du\,[(\rho_h+\rho)\ln(\rho_h+\rho) - \rho_h\ln\rho_h - \rho\ln\rho]$
  - Free energy: $\boxed{L\tilde{F}(L) = R\int du\,\{L\tilde{e}(u)\rho(u) - [(\rho_h+\rho)\ln(\rho_h+\rho) - \rho_h\ln\rho_h - \rho\ln\rho]\}}$
  - Minimize free energy with the constraint of PBC

- Lagrange multiplier

$$F[\rho_h,\rho] = R\int du\left\{L\tilde{e}(u)\rho(u) - [(\rho_h+\rho)\ln(\rho_h+\rho) - \rho_h\ln\rho_h - \rho\ln\rho] - \lambda(u)\left[\rho_h(u)+\rho(u) - \int\frac{du'}{2\pi}K(u,u')\rho(u')\right]\right\}$$

$$K(u,u') \equiv \frac{1}{i}\frac{\partial}{\partial u}\ln S(u,u')$$

$$\frac{\delta}{\delta\rho_h(u)}F[\rho_h,\rho] = \frac{\delta}{\delta\rho(u)}F[\rho_h,\rho] = 0 \quad \longrightarrow \quad \left\{\begin{array}{l}\ln\rho_h - [\ln(\rho_h+\rho))] - \lambda(u) = 0\\ L\tilde{E}(u) - [\ln(\rho_h+\rho) - \ln\rho] - \lambda(u) + \int\frac{du'}{2\pi}K(u',u)\lambda(u') = 0\end{array}\right.$$

- TBA eq.

$$\boxed{\epsilon(u) = L\tilde{e}(u) - \int\frac{du'}{2\pi}K(u',u)\ln\left[1+e^{-\epsilon(u')}\right]}$$

$$\epsilon(u) \equiv \ln[\rho_h/\rho]$$

- Minimized free energy : plug into *F* and use TBA and partial integrate

$$\boxed{E_0(L) = -\int\frac{du}{2\pi}\,\tilde{p}'(u)\ \ln\left[1+e^{-\epsilon(u)}\right]}$$

- Generalization
  - Multi-species
  - Excited states
  - Non-diagonal S-matrix

- Multi-species : with dispersion relations $(\tilde{e}_n(u),\ \tilde{p}_n(u))\,, \quad n=1,\dots,M$
- S-matrix : $S_{n,m}(u,u')$
- TBA eq. $$\boxed{\epsilon_n(u) = L\tilde{e}_n(u) - \sum_{m=1}^{M} \int \frac{du'}{2\pi} K_{nm}(u',u) \ln\left[1+e^{-\epsilon_m(u')}\right]}$$ $$K_{nm}(u,u') \equiv \frac{1}{i}\frac{\partial}{\partial u} \ln S_{nm}(u,u')$$
- Ground-state energy : $$\boxed{E_0(L) = -\sum_{n=1}^{M} \int \frac{du}{2\pi}\, \tilde{p}_n'(u)\, \ln\left[1+e^{-\epsilon_n(u)}\right]}$$

- Excited states : partial integrate $$E_0(L) = \int \frac{du}{2\pi}\, \tilde{p}(u)\, \partial_u \ln\left[1+e^{-\epsilon(u)}\right]$$ $$\epsilon(u) = L\tilde{e}(u) + \int \frac{du'}{2\pi i} \ln S(u',u)\partial_{u'} \ln\left[1+e^{-\epsilon(u')}\right]$$
- If $\ln\left[1+e^{-\epsilon(u_j)}\right]=0$, deform the integral contour and residue integrate

Mirror momentum

Physical energy

Dorey,Tateo (1996)

$$E(L) = -\sum_j i\tilde{p}(u_j) + \int \frac{du}{2\pi}\, \tilde{p}(u)\, \partial_u \ln\left[1+e^{-\epsilon(u)}\right] = \sum_j e(u_j) - \int \frac{du}{2\pi}\, \tilde{p}'(u)\, \ln\left[1+e^{-\epsilon(u)}\right]$$

$$\epsilon(u) = L\tilde{e}(u) + \sum_i \ln S(u_i,u) - \int \frac{du'}{2\pi} K(u',u) \ln\left[1+e^{-\epsilon(u')}\right]$$

- Multi-species excited states :

$$E(L) = \sum_i e_{n_i}(u_i) - \sum_{n=1}^{M} \int \frac{du}{2\pi}\, \tilde{p}'_n(u)\ \ln\left[1 + e^{-\epsilon_n(u)}\right]$$

$$\epsilon_n(u) = L\tilde{e}_n(u) + \sum_i \ln S_{n_i,n}(u_i,u) - \sum_{m=1}^{M} \int \frac{du'}{2\pi} K_{nm}(u',u) \ln\left[1 + e^{-\epsilon_m(u')}\right]$$

- Non-diagonal S-matrix :
- Diagonalize the transfer matrix to derive “Bethe-Yang” or “asymptotic Bethe” equations
- Interpret these as PBC conditions :
  - “physical” (momentum carrying) : Bethe-Yang equation
  - “magnonic” (no momentum) particles : Bethe ansatz equations
- Read off “effective” diagonal S-matrices
- Apply TBA equations derived already
- (ex) su(2) S-matrix

$$e^{ip(\theta_j)L} \prod_{n=1}^{N} \underset{\overset{\uparrow}{S_{\mathrm{pp}}}}{a(\theta_j - \theta_n)} \prod_{k=1}^{M} \underset{\overset{\uparrow}{S_{\mathrm{pm}}}}{\frac{a(u_k - \theta_j)}{b(u_k - \theta_j)}} = 1 \qquad \prod_{n=1}^{N} \underset{\overset{\uparrow}{S_{\mathrm{mp}}}}{\frac{b(u_k - \theta_n)}{a(u_k - \theta_n)}} \prod_{j\neq k, j=1}^{M} \underset{\overset{\uparrow}{S_{\mathrm{mm}}}}{\frac{u_k - u_j + i}{u_k - u_j - i}} = 1$$

Diagonal S-matrix for AdS/CFT

$$1 = \prod_{k=1}^{K_2} \overbrace{\boxed{\frac{u_{2j}-u_{2k}-i}{u_{2j}-u_{2k}+i}}}^{S_{22}} \prod_{k=1}^{K_3+K_1} \overbrace{\boxed{\frac{u_{2j}-u_{3k}+\frac{i}{2}}{u_{2j}-u_{3k}-\frac{i}{2}}}}^{S_{23}}$$

$$1 = \underbrace{\boxed{\prod_{k=1}^{K_2} \frac{u_{3j}-u_{2k}+\frac{i}{2}}{u_{3j}-u_{2k}-\frac{i}{2}}}}_{S_{32}} \prod_{k=1}^{K_4} \underbrace{\boxed{\frac{x_{3j}-x^+_{4k}}{x_{3j}-x^-_{4k}}}}_{S_{34}}$$

$$\left(\frac{x^+_{4j}}{x^-_{4j}}\right)^{L'} = \prod_{k=1}^{K_4} \underbrace{\boxed{\sigma^2(x_{4j},x_{4k})\frac{u_{4j}-u_{4k}+i}{u_{4j}-u_{4k}-i}}}_{S_{44}}$$

$$\times \prod_{k=1}^{K_3+K_1} \underbrace{\boxed{\frac{x^-_{4j}-x_{3k}}{x^+_{4j}-x_{3k}}}}_{S_{43}} \prod_{k=1}^{K_5+K_7} \frac{x^-_{4j}-x_{5k}}{x^+_{4j}-x_{5k}}$$

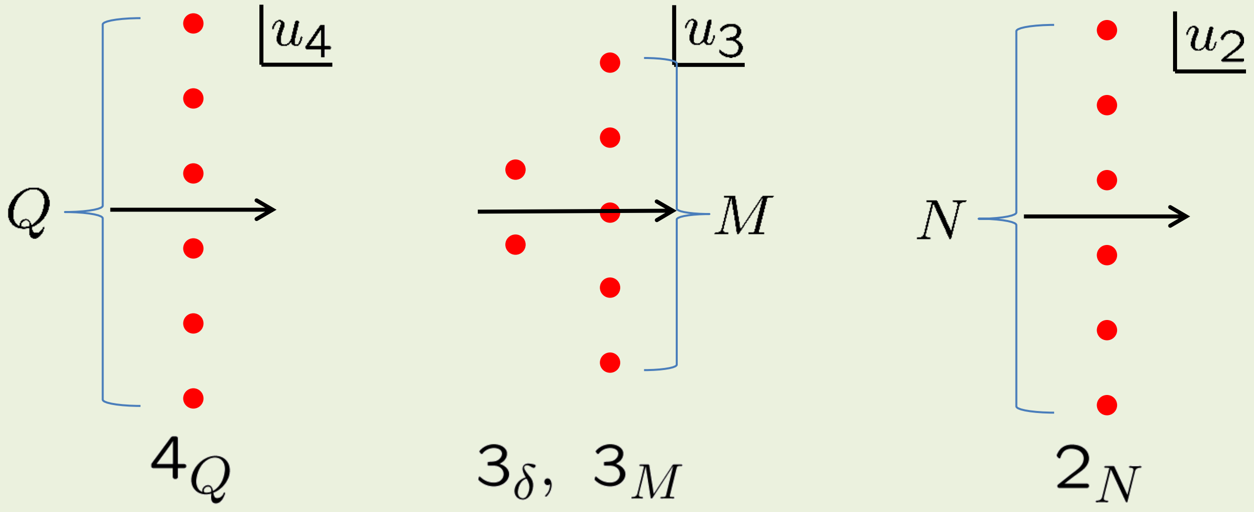
$$1 = \prod_{k=1}^{K_6} \frac{u_{5j}-u_{6k}+\frac{i}{2}}{u_{5j}-u_{6k}-\frac{i}{2}} \prod_{k=1}^{K_4} \frac{x_{5j}-x^+_{4k}}{x_{5j}-x^-_{4k}}$$

$$1 = \prod_{k=1}^{K_6} \frac{u_{6j}-u_{6k}-i}{u_{6j}-u_{6k}+i} \prod_{k=1}^{K_5+K_7} \frac{u_{6j}-u_{5k}+\frac{i}{2}}{u_{6j}-u_{5k}-\frac{i}{2}}$$

$$\boxed{\begin{aligned}
1 &= \prod_{k=1}^{M_2} S_{22}(x_{2j},x_{2k}) \prod_{k=1}^{M_3} S_{23}(x_{2j},x_{3k}) \\
1 &= \prod_{k=1}^{M_2} S_{32}(x_{3j},x_{2k}) \prod_{k=1}^{M_4} S_{34}(x_{3j},x_{4k}) \\
1 &= e^{ip_jR} \prod_{k=1}^{M_4} S_{44}(x_{4j},x_{4k}) \prod_{k=1}^{M_3} S_{43}(x_{4j},x_{3k}) \prod_{k=1}^{M_5} S_{43}(x_{4j},x_{5k}) \\
1 &= \prod_{k=1}^{M_6} S_{32}(x_{5j},x_{6k}) \prod_{k=1}^{M_4} S_{34}(x_{5j},x_{4k}) \\
1 &= \prod_{k=1}^{M_6} S_{22}(x_{6j},x_{6k}) \prod_{k=1}^{M_5} S_{23}(x_{6j},x_{5k})
\end{aligned}}$$

String hypothesis

- AdS/CFT contains infinite # of bound states and need their ABAEs
- The bound states belong to higher dimensional representation of su(2|2) and their S-matrices can not be determined by su(2|2) but need "yangian" symmetry
- Bypassing derivation ABAE for the bound states, one can find the "diagonal" S-matrices by studying the string solutions by a similar logic of su(2) case
- Classes of strings $4_Q, \ 3_\delta, \ 3_M, \ 2_N$



- Diagonal S-matrices

$$
\begin{aligned}
S_{44}^{(QQ')} &= \sigma_{QQ'} E_{QQ'} \\
S_{43}^{(QM)} &= \frac{x(u_{-Q}) - x(v_M)}{x(u_Q) - x(v_M)} \, \frac{x(u_{-Q}) - x(v_{-M})}{x(u_Q) - x(v_{-M})} \, \frac{x(u_Q)}{x(u_{-Q})} \prod_{j=1}^{M-1} e_{M-Q-2j} \\
S_{43}^{(Q\delta)} &= \frac{x(u_{-Q}) - x(v)^{\delta}}{x(u_Q) - x(v)^{\delta}} \sqrt{\frac{x(u_Q)}{x(u_{-Q})}} \\
S_{33}^{(MM')} &= S_{22}^{(MM')^{-1}} = E_{MM'} \\
S_{33}^{(M\delta)} &= S_{23}^{(M\delta)} = e_M
\end{aligned}
$$

$$e_n(u) \equiv \frac{u + in/2g}{u - in/2g}$$

$$E_{n,m} = e_{|n-m|} e^2_{|n-m|+2} \cdots e^2_{n+m-2} e_{n+m}$$

$$x(u_M) + \frac{1}{x(u_M)} = u_M, \quad x^+(u_M) + \frac{1}{x^+(u_M)} - x^-(u_M) - \frac{1}{x^-(u_M)} = \frac{iM}{g}$$

- Asymptotic BAE for these strings can be constructed straightforwardly since the scatterings are diagonal and TBA can be derived accordingly

TBA for AdS/CFT

- Thermodynamic BAE

Arutyunov,Frolov; Bombardelli,Fioravanti,Tateo; Gromov,Kazakov,Kozak,Vieira (2009)

$$\begin{array}{rcl} \epsilon_4^{(Q)} & = & L\tilde{e}_Q - L_4^{(Q')} \star K_{44}^{(Q'Q)} - L_3^{(M)} \star K_{34}^{(MQ)} - L_3^{(\delta)} \star K_{34}^{(\delta Q)} \\ \epsilon_3^{(M)} & = & -L_4^{(Q)} \star K_{43}^{(QM)} - L_3^{(M')} \star K_{33}^{(M'M)} - L_3^{(\delta)} \star K_{33}^{(\delta M)} \\ \epsilon_2^{(N)} & = & L_2^{(N')} \star K_{22}^{(N'N)} - L_3^{(\delta)} \star K_{32}^{(\delta N)} \\ \epsilon_3^{(\delta)} & = & -L_4^{(Q)} \star K_{43}^{(Q\delta)} - L_3^{(M)} \star K_{33}^{(M\delta)} - L_2^{(N)} \star K_{23}^{(N\delta)} \end{array}$$

$$A \star K(u) = \int \frac{du'}{2\pi} A(u')K(u',u)$$

$$K_{ab}^{(nm)}(u,u') \equiv -i\partial_u \ln S_{ab}^{(nm)}(u,u')$$

- Physical dispersion relation
- Mirror one

$$e_n(p) = \sqrt{n^2 + 16g^2 \sin^2 \frac{p}{2}} \qquad (e,p) = (i\tilde{p}, -i\tilde{e})$$

$$\tilde{e}_n(\tilde{p}) = 2 \sinh^{-1}\left(\frac{1}{4g}\sqrt{\tilde{p}^2 + n^2}\right)$$

- Finite-size energy

$$E_0(L) = -\sum_{Q=1}^{\infty} \int \frac{du}{2\pi}\, \tilde{p}'_Q \, \ln\left(1 + e^{-\epsilon_4^{(Q)}}\right)$$

- Excited TBA by analytic continuation or by “Y-system”

Universal kernel

- One can introduce kernel which connects only nearest neighbors

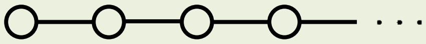
$$s\mathbb{I}_{MN} = \delta_{MN} - (K+1)^{-1}_{MN}, \qquad s(u) = \frac{g}{2\cosh g\pi u}$$

Incidence matrix of a Dynkin diagram

$$K \equiv \frac{1}{i}\frac{\partial}{\partial u}\ln S$$

(ex) bound-state S-matrix of su(2)-invariant theory

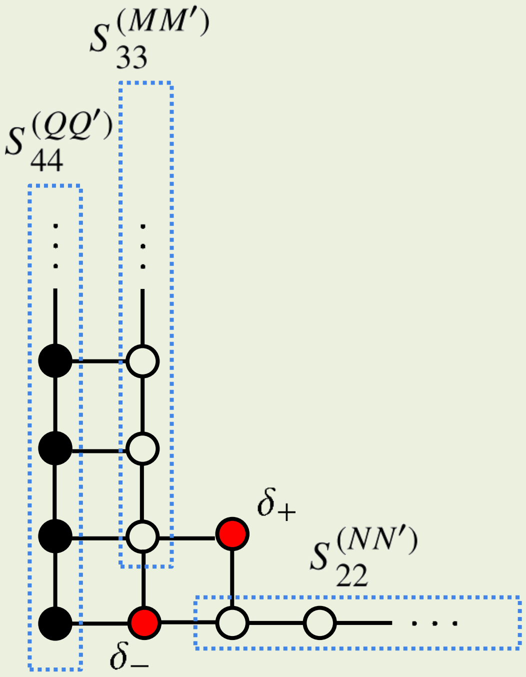
$$S^{(nm)}(u-v) = E_{nm}(u-v) = e_{|n-m|}e^2_{|n-m|+2}\cdots e^2_{n+m-2}e_{n+m}(u-v)$$



$$\mathbb{I}_{MN} = \delta_{M,N-1} + \delta_{M,N+1}$$

- universal kernel for AdS/CFT (2d Dynkin diagram)

$$S^{(nm)}_{ab}(u,v) \to K^{(nm)}_{ab}$$



- Define Y-functions :

$$Y_{Q,0} = e^{-\epsilon_4^{(Q)}}, \; Y_{M+1,1} = e^{-\epsilon_3^{(M)}}, \; Y_{1,N+1} = e^{\epsilon_2^{(N)}}, \; Y_{1,1} = e^{\epsilon_3^{(\delta=-)}}, \; Y_{2,2} = e^{\epsilon_3^{(\delta=+)}}$$

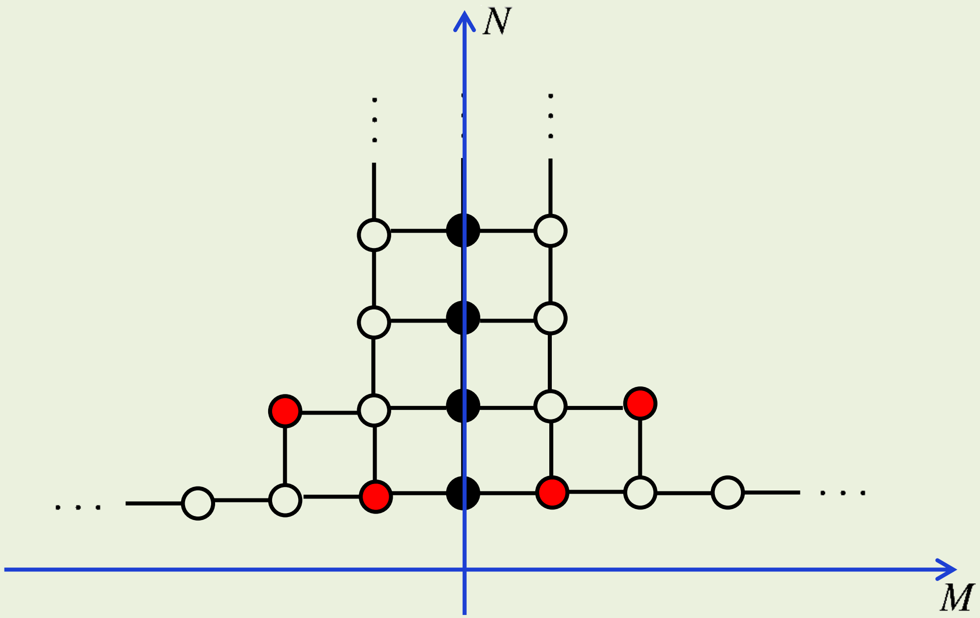
- Add another side of su(2|2) S-matrix

$$Y_{Q,0} = e^{-\epsilon_4^{(Q)}}, \; Y_{M+1,-1} = e^{-\epsilon_3^{(M)}}, \; Y_{1,-(N+1)} = e^{\epsilon_2^{(N)}}, \; Y_{1,-1} = e^{\epsilon_3^{(\delta=-)}}, \; Y_{2,-2} = e^{\epsilon_3^{(\delta=+)}}$$

- TBA with universal kernel

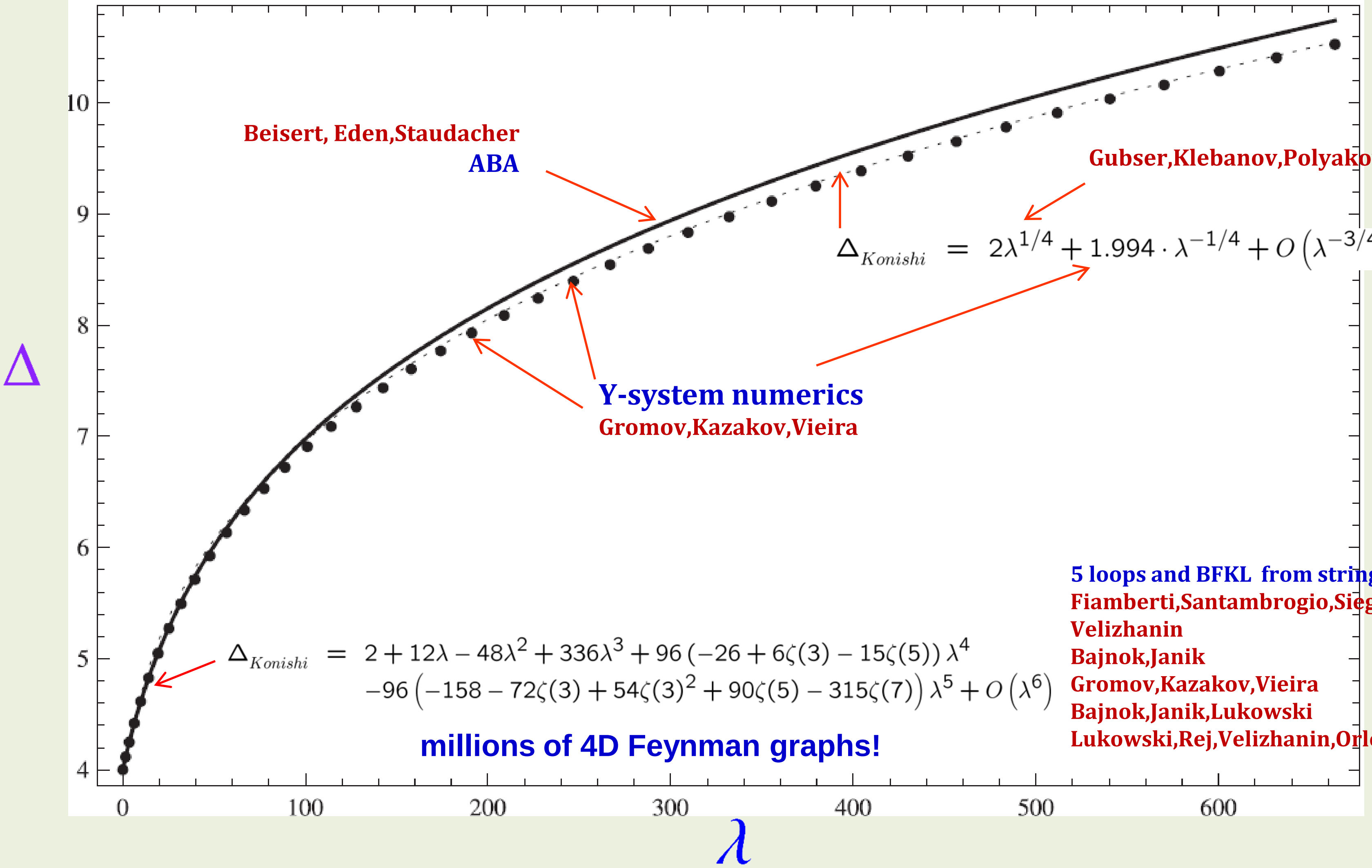
$$\ln Y_{N,M} = s\star\Big[\ln(1+Y_{N,M+1}) + \ln(1+Y_{N,M-1})\Big] - s\star\Big[\ln(1+Y^{-1}_{N+1,M}) + \ln(1+Y^{-1}_{N-1,M})\Big]$$

  - "deriving term" can be absorbed into boundary condition



- Numerics for sl(2) Konishi

$$\mathrm{Tr}\,[\mathcal{D}, Z]^2$$



Beisert, Eden,Staudacher
ABA

Gubser,Klebanov,Polyakov

$$\Delta_{Konishi} = 2\lambda^{1/4} + 1.994 \cdot \lambda^{-1/4} + O\left(\lambda^{-3/4}\right)$$

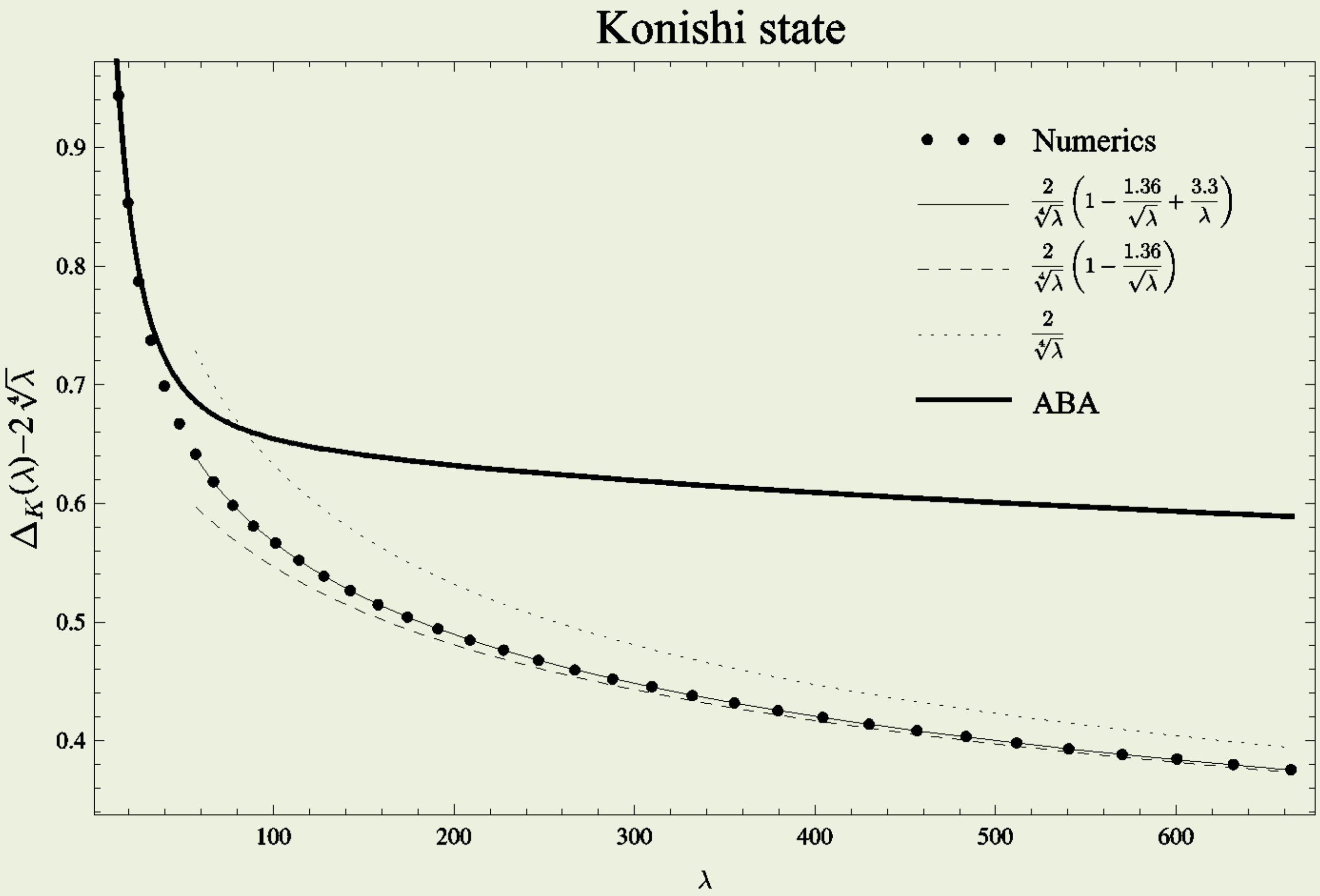
Y-system numerics
Gromov,Kazakov,Vieira

$$\Delta_{Konishi} = 2 + 12\lambda - 48\lambda^2 + 336\lambda^3 + 96\left(-26 + 6\zeta(3) - 15\zeta(5)\right)\lambda^4$$
$$-96\left(-158 - 72\zeta(3) + 54\zeta(3)^2 + 90\zeta(5) - 315\zeta(7)\right)\lambda^5 + O\left(\lambda^6\right)$$

millions of 4D Feynman graphs!

5 loops and BFKL from string
Fiamberti,Santambrogio,Sieg,Zanon
Velizhanin
Bajnok,Janik
Gromov,Kazakov,Vieira
Bajnok,Janik,Lukowski
Lukowski,Rej,Velizhanin,Orlova

- Difference from GKP scaling



Gromov,Kazakov,Vieira (2009); Frolov (2010)

Y-system

Al. Zamolodchikov (1991)

- Thermodynamic BAE is integral equations
- Difficult to generalize to excited states
- Y-system is a system of functional equations which can be derived from TBA
  - (ex) su(2)

$$\ln Y_N(u) = \sum_M \mathbb{I}_{NM}\, s \star [\ln(1+Y_M)]\,(u)$$

Fourier transform

$$\overline{\ln Y_N}(k) = \sum_M \mathbb{I}_{NM}\, \bar{s}\, \overline{[\ln(1+Y_M)]}(k), \qquad \bar{f}(k) = \int \frac{du}{\sqrt{2\pi}}\, f(u)\, e^{iku}$$

$$2\cosh\frac{k}{2g}\,\overline{\ln Y_N}(k) = \sum_M \mathbb{I}_{NM}\, \overline{[\ln(1+Y_M)]}(k), \qquad \bar{s}(k) = \frac{1}{2\cosh\frac{k}{2g}}$$

$$\ln Y_N\left(u+\frac{i}{2g}\right) + \ln Y_N\left(u-\frac{i}{2g}\right) = \sum_M \mathbb{I}_{NM}\, [\ln(1+Y_M)]\,(u), \qquad Y_N^{\pm}(u) \equiv Y_N\left(u \pm \frac{i}{2g}\right)$$

$$Y_N^+(u)\, Y_N^-(u) = \prod_M [1+Y_M(u)]^{\mathbb{I}_{NM}} = [1+Y_{N-1}(u)]\,[1+Y_{N+1}(u)]$$

  - AdS/CFT

$$\boxed{Y^+_{N,M}Y^-_{N,M} = \frac{(1+Y_{N,M+1})(1+Y_{N,M-1})}{(1+Y^{-1}_{N+1,M})(1+Y^{-1}_{N-1,M})}}$$

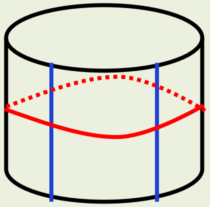
Gromov,Kazakov,Vieira (2009)

- Solutions for the Y-system are not unique and include all the excited states

TBA in IR limit : Luscher correction

- Analytic analysis is possible when $L\tilde{e}_n(u)$ is large
- Consider two-particle excitation for one-particle species theory
  - TBA eq. $\epsilon(u) = L\tilde{e}(u) + \ln S(u_1,u) + \ln S(u_2,u) - \int \frac{du'}{2\pi} K(u',u) \ln\left[1+e^{-\epsilon(u')}\right]$
  - Constraint eq. $1 + e^{-\epsilon(u_i)} = 0$
  - Energy $E(L) = e(u_1) + e(u_2) - \int \frac{du}{2\pi}\, \tilde{p}'(u)\, \ln\left[1+e^{-\epsilon(u)}\right]$
- In the leading order
  - In the limit $L\tilde{e}(u) \gg 1 \;\rightarrow\; \epsilon(u) \approx L\tilde{e}(u) + \ln S(u_1,u) + \ln S(u_2,u)$
  - Bethe-Yang eq. $e^{-\epsilon(u_1)} = -e^{-ip(u_1)L} S(u_2,u_1) = -1$
  - Finite-size correction for energy

$$\begin{aligned} E &= e(u_1) + e(u_2) - \int \frac{du}{2\pi} \tilde{p}' e^{-L\tilde{e}(u)} \frac{1}{S(u_1,u)S(u_2,u)} \\ &= e(u_1) + e(e_2) - \int \frac{dq}{2\pi} e^{-L\tilde{e}(q)} S(u,u_1)S(u,u_2), \qquad q \equiv \tilde{p}(u) \end{aligned}$$



- In the next order : keeping an exponentially small

$$\epsilon(u) = L\tilde{e}(u) + \ln S(u_1, u) + \ln S(u_2, u) - \int \frac{du'}{2\pi} K(u', u) \ln\left[1 + e^{-L\tilde{e}(u')}\right]$$

  - Impose the constraint $1 + e^{-\epsilon(u_i)} = 0$

$$\begin{aligned} 0 &= \underbrace{\log\{e^{iLp_1} S(u_2, u_1)\}}_{BY_1} + \underbrace{\int \frac{du}{2\pi i} (\partial_u S(u, u_1)) S(u, u_2) e^{-L\tilde{e}(u)}}_{\Phi_1} \\ 0 &= \underbrace{\log\{e^{iLp_2} S(u_1, u_2)\}}_{BY_2} + \underbrace{\int \frac{du}{2\pi i} S(u, u_1)(\partial_u S(u, u_2)) e^{-L\tilde{e}(u)}}_{\Phi_2} \end{aligned}$$

  - Energy

$$E = e(u_1) + e(u_2) + e'(p_1)\delta p_1 + e'(p_2)\delta p_2 - \int \frac{dq}{2\pi} e^{-L\tilde{e}} S(u, u_1) S(u, u_2)$$

with

$$\begin{aligned} \frac{\partial BY_1}{\partial p_1}\delta p_1 + \frac{\partial BY_1}{\partial p_2}\delta p_2 + \Phi_1 &= 0 \\ \frac{\partial BY_2}{\partial p_1}\delta p_1 + \frac{\partial BY_2}{\partial p_2}\delta p_2 + \Phi_2 &= 0 \end{aligned}$$

Wrapping correction for 4-loop Konishi

- $\delta\triangle = \triangle_{\text{Pert.}} - \triangle_{\text{BAE}} = (324+864\zeta(3)-1440\zeta(5))g^8 + \mathcal{O}(g^{10})$ **Bajnok, Janik (2008)**
- Luscher formula : (*L*=4) [μterm vanishes]

$$E(L) = \sum_{k=1,2} e_a(p_k) - \int_{-\infty}^{\infty} \frac{dq}{2\pi} \sum_{a_1,a_2} (-1)^F \left[S_{a_1 a}^{a_2 a}(q,p_1) S_{a_2 a}^{a_1 a}(q,p_2) \right] e^{-L\tilde{e}_{a_1}(q)}$$

  - Exponential factor using the mirror dispersion relation

$$\tilde{e}_n(\tilde{p}) = 2\sinh^{-1}\left(\frac{1}{4g}\sqrt{\tilde{p}^2+n^2}\right) \ : \ e^{-2L\sinh^{-1}\frac{\sqrt{n^2+q^2}}{4g}} \to \frac{4^L g^{2L}}{(n^2+q^2)^L} \sim \mathcal{O}(g^8)$$

  - All the bound states contribute to the same order and one needs the matrix element and dressing factor for these in the mirror space (sl(2) grading)
  - After some algebras, the integrand becomes

$$-\int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{147456Q^2(3q^3+3Q^2-4)^2}{(q^2+Q^2)^4(9q^4+6[3(Q-2)Q+2]q^2+[3(Q-2)Q+4]^2)} \frac{1}{9q^4+6[3(Q+2)Q+2]q^2+[3(Q+2)Q+4]^2}$$

  - Residue integrals

$$\sum_{Q=1}^{\infty}\left\{-\frac{\text{num}(Q)}{(9Q^4-3Q^2+1)^4(27Q^6-27Q^4+36Q^2+16)} + \frac{864}{Q^3} - \frac{1140}{Q^5}\right\}$$

$$\text{num}(Q) = 7776Q(19683Q^{18} - 78732Q^{16} + 150903Q^{14} - 134865Q^{12} + 1458Q^{10} + 48357Q^8 - 13311Q^6 - 1053Q^4 + 369Q^2 - 10)$$

$$\delta E = 324 + 864\zeta(3) - 1140\zeta(5)$$

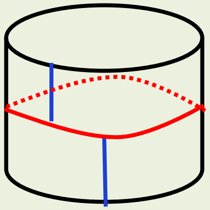
Classical string with finite J

- Energe correction for a single Giant Magnon $J \gg g \gg 1$ **Janik,Lukowski (2007)**

$$\delta E \approx -16 g \sin^3 \frac{p}{2} \ \exp\left[-\left(\frac{J}{2g\sin\frac{p}{2}}+2\right)\right]+\dots$$

- In this limit, the leading contribution comes from μ term
- μ term Luscher formula $\sin^2 \frac{p}{2} \quad \frac{8i}{g\sin^3\frac{p}{2}} \ e^{ip} \ \sigma^2(q^*,p) \quad e^{-\frac{L}{2g\sin^2\frac{p}{2}}}$

$$\delta E \approx -i\left[1-\frac{\epsilon'(p)}{\epsilon'(q^*)}\right]\cdot \underset{q=q^*}{\text{res}} \sum_b S_{ba}^{ba}(q,p)\cdot e^{-iLq^*}$$



  - Leading contribution from elementary GM
  - Pole arises at $q^* = \frac{-i}{2g\sin\frac{p}{2}}$
  - Using AFS dressing factor $\sigma^2(q^*,p) = -2g^2 \ \sin^4\frac{p}{2} \ e^{-ip}$
  - Combining together

$$\delta E_{\text{Luscher}} \approx -16 g \sin^3 \frac{p}{2} \ e^{-\frac{J}{2g\sin\frac{p}{2}}}$$

Y-system and Hirota equation

Gromov,Kazakov,Vieira (2009)

- Hirota eq. for T(ransfer matrix)-system
  - Y-system $$Y^+_{a,s}Y^-_{a,s} = \frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+Y^{-1}_{a+1,s})(1+Y^{-1}_{a-1,s})}$$
  - Can be satisfied if $Y_{a,s} = \dfrac{T_{a,s+1}T_{a,s-1}}{T_{a+1,s}T_{a-1,s}}$ with $T^+_{a,s}T^-_{a,s} = T_{a+1,s}T_{a-1,s}+T_{a,s+1}T_{a,s-1}$

$$E(L) = \sum_j e_1(u_{4j}) - \sum_{a=1}^{\infty} \int \frac{du}{2\pi}\, \tilde{p}'_a\, \ln\left(1+Y_{a,0}(u)\right)$$

- IR limit $L\tilde{e}_n(u) \gg 1$

$$Y_{a,0} \approx e^{-L\tilde{e}_a(u)} = \left(\frac{x^{[-a]}}{x^{[a]}}\right)^L \ll 1, \quad Y_{a,s\neq 0} \to \text{const.}\,; \quad x^{[k]}(u) \equiv x(u+ik/2)$$

$$Y^+_{a,0}Y^-_{a,0} = \frac{(1+Y_{a,1})(1+Y_{a,-1})}{(1+Y^{-1}_{a+1,0})(1+Y^{-1}_{a-1,0})} \approx \frac{(1+Y_{a,1})(1+Y_{a,-1})}{Y^{-1}_{a+1,0}\;Y^{-1}_{a-1,0}} \to \frac{Y^+_{a,0}Y^-_{a,0}}{Y_{a+1,0}Y_{a-1,0}} = (1+Y_{a,1})(1+Y_{a,-1}) = \frac{T^+_{a,1}T^-_{a,1}}{T_{a+1,1}T_{a-1,1}}\,\frac{T^+_{a,-1}T^-_{a,-1}}{T_{a+1,-1}T_{a-1,-1}}$$

$$\downarrow$$

$$Y_{a,0} = T_{a,1}T_{a,-1}\left(\frac{x^{[-a]}}{x^{[a]}}\right)^L \cdot \frac{\phi^{[-a]}}{\phi^{[a]}}$$

- Find $T_{a,\pm 1},\quad \phi$
  - $T_{1,1}$ from eigenvalue of transfer matrix

$$T_{1,1} \;=\; \frac{R^{-(+)}}{R^{-(-)}}\left[\frac{Q_2^{[-2]}Q_3^{+}}{Q_2 Q_3^{-}} - \frac{R^{-(-)}Q_3^{+}}{R^{-(+)}Q_3^{-}} + \frac{Q_2^{[2]}Q_1^{-}}{Q_2 Q_1^{+}} - \frac{B^{+(+)}Q_1^{-}}{B^{+(-)}Q_1^{+}}\right]$$

$$Q_l(u) \;=\; \prod_{j=1}^{K_l}(u-u+lj),\;\; R_l^{(\pm)}(u) = \prod_{j=1}^{K_l}\frac{x(u)-x_{lj}^{\mp}}{(x_{lj}^{\mp})^{1/2}},\;\; B_l^{(\pm)}(u) = \prod_{j=1}^{K_l}\frac{1/x(u)-x_{lj}^{\mp}}{(x_{lj}^{\mp})^{1/2}}$$

  - From the constraint equation $Y_{1,0}(u_{4j}) = -1$ which should be asymp. BAE,

$$\frac{\phi^{-}}{\phi^{+}} = \sigma^2 \frac{B^{+(+)}R^{-(-)}}{B^{-(-)}R^{+(+)}}\frac{B_1^{+}B_3^{-}}{B_1^{-}B_3^{+}}\frac{B_7^{+}B_5^{-}}{B_7^{-}B_5^{+}}$$

- Generating function for $T_{a,\pm 1}$: $\quad D = e^{-i\partial_u}$

$$\mathcal{W} \;=\; \left[1-\frac{B^{+(+)}Q_1^{-}R^{-(+)}}{B^{+(-)}Q_1^{+}R^{-(-)}}D\right]\left[1-\frac{Q_2^{[2]}Q_1^{-}R^{-(+)}}{Q_2 Q_1^{+}R^{-(-)}}D\right]^{-1}\left[1-\frac{Q_2^{[-2]}Q_3^{+}R^{-(+)}}{Q_2 Q_3^{-}R^{-(-)}}D\right]^{-1}\left[1-\frac{Q_3^{+}}{Q_3^{-}}D\right]$$

$$\mathcal{W}^{-1} \;=\; \sum_{a=0}^{\infty}(-1)^a T_{a,1}D^a$$

- Gives exactly the same result as TBA Luscher formula

Topics not covered

- AdS_4/CFT_3 duality
  - Type IIA strings on CP3 & ABJM model
- Open string attached on Giant graviton
  - Related to boundary integrability
- With less supersymmetry
  - Beta deformed theory
- Higher-point correlation functions

- Space-time scattering amplitudes