Nonlinear Integral equation for Sausage model

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with

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(J. Phys. A, “Petr Kulish memorial issue”)

Outline

1. Introduction to “integrable QFT” in 2 dimensions

2. Review of Sausage model

3. Derivation of non-liner integral equation

4. UV and IR limits

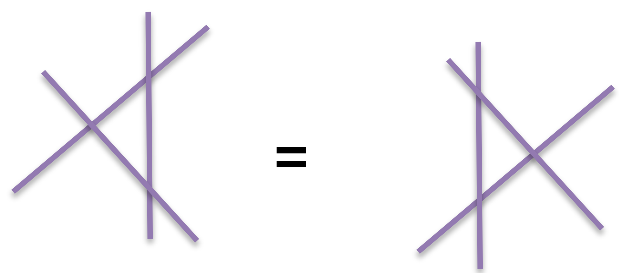
Integrable QFT in 2D

- Infinitely many conserved quantities

$$Q_n = \sum_j p_j^n = \sum_j {p'_j}^n \quad \rightarrow \quad \{p_j\} = \{p'_j\}$$

- Scattering is factorized into 2→2 exact "S-matrix" which satisfy Yang-Baxter equation

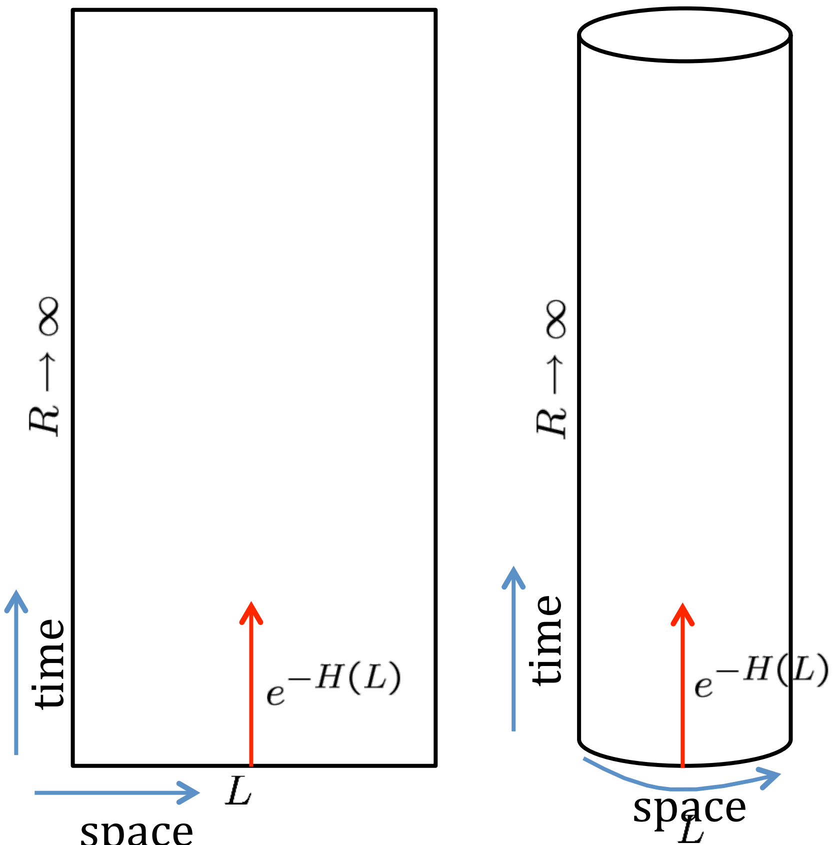
$$\mathbf{S}_{12}(p_1,p_2)\,\mathbf{S}_{13}(p_1,p_3)\,\mathbf{S}_{23}(p_2,p_3) = \mathbf{S}_{23}(p_2,p_3)\,\mathbf{S}_{13}(p_1,p_3)\,\mathbf{S}_{12}(p_1,p_2)$$



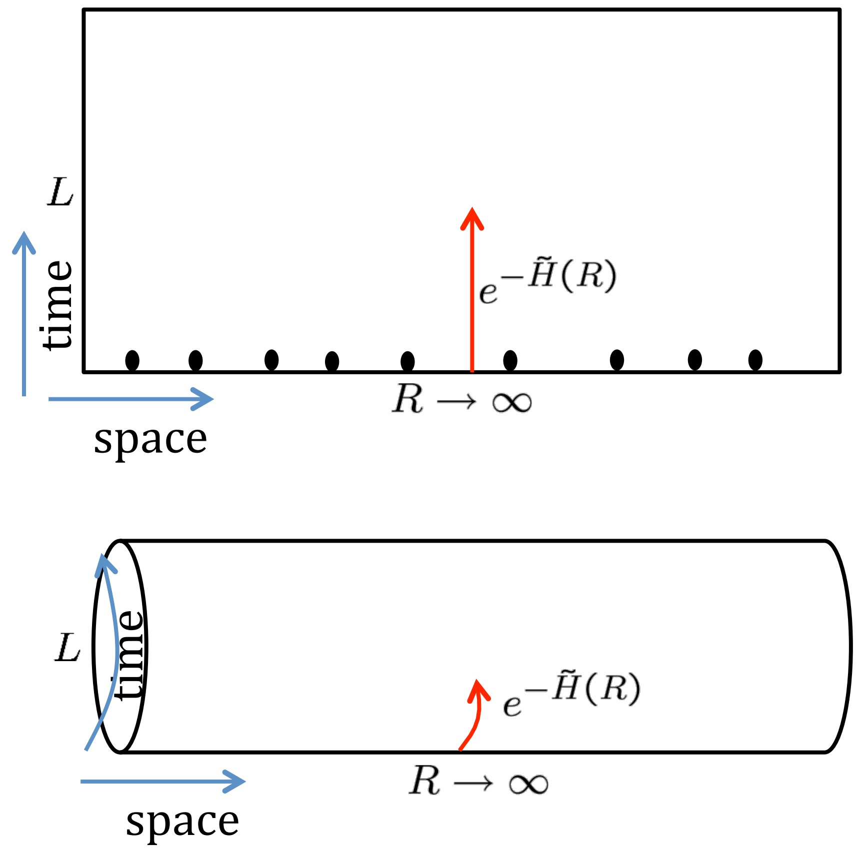
- S-matrix is used to compute physical quantities
  - (ex) Finite-size effect

Thermodynamic Bethe Ansatz (Al. B. Zamolodchikov)

Physical space

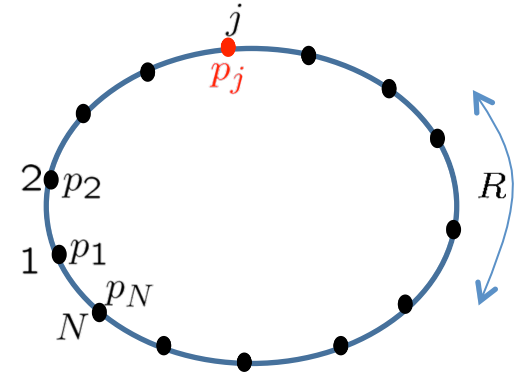


Mirror space



Channel Duality

- Mirror channel
  - Scatterings between asymptotic particles are valid since $R \to \infty$
  - N-particles in a box of length R
  - Bethe-Yang equation $e^{ip_j R} \prod_{k \neq j,1}^{N} S(p_j, p_k) = 1$
  - Partition function $\tilde{Z}(R,L) = \mathrm{Tr}\left[e^{-L\tilde{H}(R)}\right]$



- Physical channel
  - Partition function $Z(L,R) = \mathrm{Tr}\left[e^{-RH(L)}\right] \approx e^{-RE_0(L)}$ as $R \to \infty$

$$\tilde{Z}(R,L) = Z(L,R) \quad \to \quad E_0(L) = -\frac{1}{R}\ln \tilde{Z}(R,L) = \frac{L}{R}\tilde{\mathcal{F}}(L)$$

Free energy with temperature $T = \frac{1}{L}$

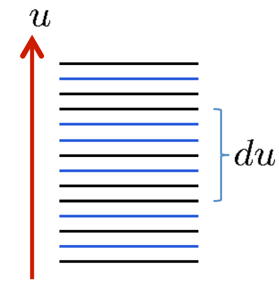
  - Computing free energy in the mirror space $\tilde{\mathcal{F}}(L) = \tilde{E} - TS$

- Mirror free energy with dispersion relation $(e(u),\ p(u))$ ← "rapidity"
  - Log of Bethe-Yang equation :

$$p_j(u) - \frac{i}{R}\sum_{k\neq j,1}^{N} \ln S(p_j,p_k) = 2\pi\frac{n_j}{R} \rightarrow p_j(u) + \int du'\rho(u')\frac{1}{i}\ln S(p(u),p(u')) = 2\pi\frac{n_j}{R}$$

(any integer: n_j)

$$\rightarrow \frac{dp}{du} + \int du'\rho(u')\underbrace{\frac{1}{i}\frac{\partial}{\partial u}\ln S(u,u')}_{K(u,u')} = 2\pi[\rho_h(u)+\rho(u)] \qquad N,\ R\rightarrow\infty$$



  - Energy

$$\widetilde{E} = \sum_{j=1}^{N} e(u_j) = R\int du\ \rho(u)\ e(u), \quad \rho(u) = \frac{1}{R}\frac{dn}{du}, \quad \rho_h(u) = \frac{1}{R}\frac{dn_h}{du},$$

    - n = density of particles,
    - dn = density of particles with rapidity between u and $u+du$
    - n_h = density of unoccupied ('holes') states
  - Entropy : log of no. of possibility $\mathcal{S} = R\int du\,[(\rho_h+\rho)\ln(\rho_h+\rho) - \rho_h\ln\rho_h - \rho\ln\rho]$
  - Free energy:

$$L\widetilde{F}(L) = R\int du\,\{Le(u)\rho(u) - [(\rho_h+\rho)\ln(\rho_h+\rho) - \rho_h\ln\rho_h - \rho\ln\rho]\}$$

- Minimize free energy with the constraint of PBC
  - TBA Eq.

$$\epsilon(u) = Le(u) - \int\frac{du'}{2\pi}K(u',u)\ln\left[1+e^{-\epsilon(u')}\right] \qquad \epsilon(u)\equiv\ln\frac{\rho_h}{\rho}$$

$$K\star\ln(1+e^{-\epsilon})$$

  - Ground-state Energy

$$E_0(L) = -\int\frac{du}{2\pi}\,p'(u)\ \ln\left[1+e^{-\epsilon(u)}\right]$$

- Multi-species
  - S-matrix :

$$(e_n(u),\ p_n(u)), \quad n = 1, \ldots, M$$

$$S_{n,m}(u,u') \qquad K_{nm}(u,u') \equiv \frac{1}{i}\frac{\partial}{\partial u} \ln S_{nm}(u,u')$$

  - TBA eq.

$$\epsilon_n(u) = L e_n(u) - \sum_{m=1}^{M} \int \frac{du'}{2\pi} K_{nm}(u',u) \ln\left[1 + e^{-\epsilon_m(u')}\right]$$

  - Ground-state energy :

$$E_0(L) = -\sum_{n=1}^{M} \int \frac{du}{2\pi}\, p_n'(u)\ \ln\left[1 + e^{-\epsilon_n(u)}\right]$$

- Non-diagonal S-matrix
  - Diagonalize the transfer matrix to derive "Bethe-Yang"
  - "physical" (momentum carrying) : Bethe-Yang equation
  - "magnonic" (no momentum) particles : Bethe ansatz equations

$$e^{ip_j R} \underbrace{\prod_{k \neq j,1}^{N} S(p_j, p_k)}_{\text{Transfer matrix}} = 1$$

- Finite-size Effect

$$c_{\text{eff}} = -\frac{6 L E_0(L)}{\pi}$$

  - Interpolates from UV CFT to IR (massive or IR CFT)

Nonlinear Integral Eq. (NLIE)

[Klümper,Pearce;Destri,deVega]

- TBA includes often infinite # of equations
- Based on an integrable lattice model with Bethe ansatz
  - Sum over vacuum Bethe roots lead to NLIE
  - Continuum limit (a→ 0) with lattice size $N \to \infty$

$$L = Na, \quad m = \frac{1}{a} e^{-\Lambda}$$

  - Only a few equations
  - Easily generalized to excited states

- Connection to underlying QFT is indirect
  - S-matrix elements of two-particle excited states

"Zoo" of Integrable QFTs

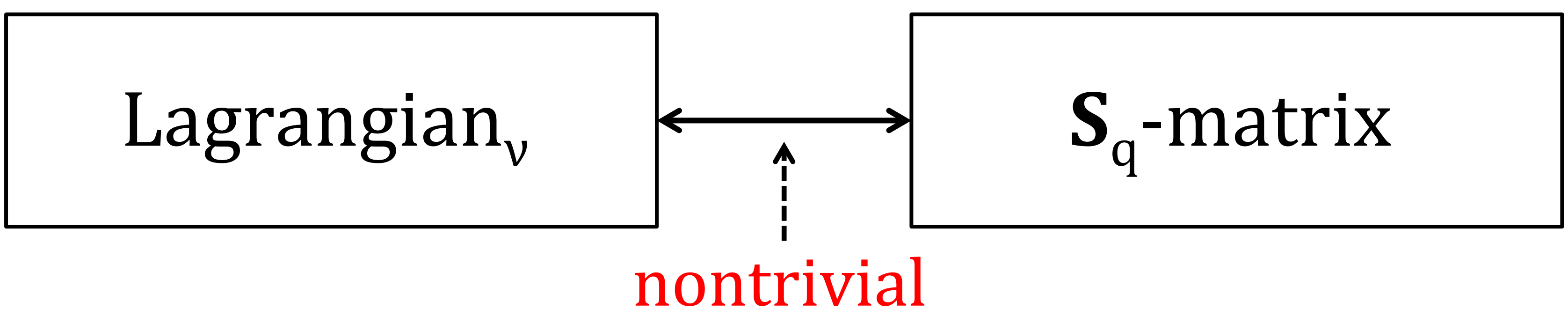
classes	(ex)	$\mathcal{L}$	spectrum	**S**
Affine Toda theories	Sinh-Gordon	✓	fund. fields	diago.
Nonlinear σ-models	O(3), AdS/CFT	✓	Reps	non-diago.
Perturbed CFTs	RSOS	✗	kinks	non-diago.

More?

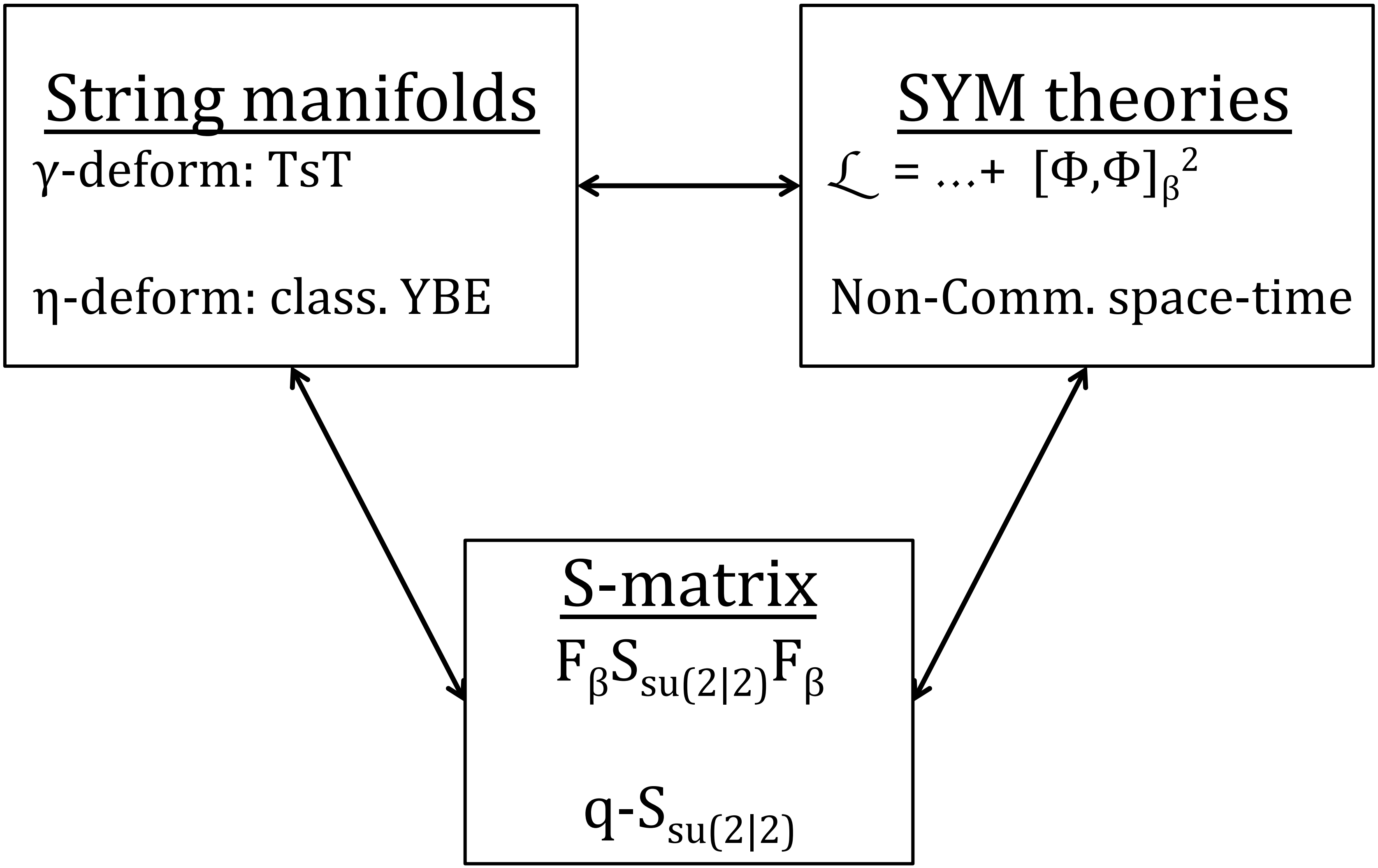
Integrable deformations

- Deform S-matrix
  - Drinfeld-Reshetikhin twist
  - Quantum Group

- Deform Lagrangian
  - Classical Yang-Baxter algebra [Klimcik]
  - Discrete symmetries of target mainfolds



Ex. deformed AdS/CFT

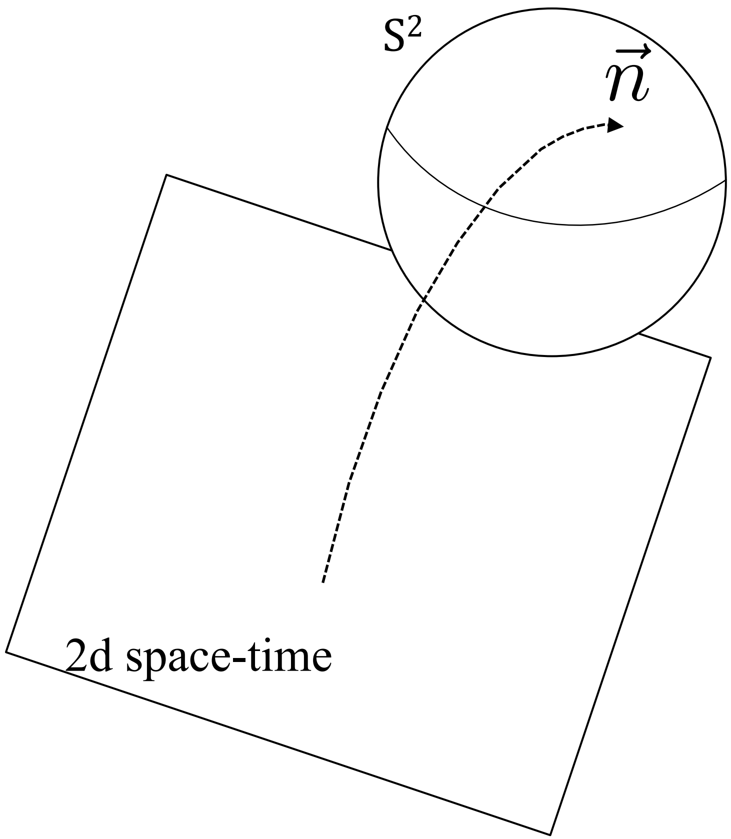


Review on Sausage model

- Fateev, Onofri, Zamolodchikov (1993)

- Integrable deformation of O(3) σ-model

O(3) nonlinear σ-model



$$|\vec{n}| = 1$$

$$\begin{aligned} \mathcal{A} &= \frac{1}{2g}\int d^2x\,(\partial_\mu \vec{n})^2 + i\vartheta T \\ T &= \frac{1}{8\pi}\epsilon^{\mu\nu}\vec{n}\cdot(\partial_\mu \vec{n}\times\partial_\nu \vec{n}) \end{aligned}$$

Haldane Conjecture: O(3) SM is equivalent to spin-s anti-ferro Heisenberg model in large s limit

s=integer ←→ $\vartheta = 0$

s=half-integer ←→ $\vartheta = \pi$

$$H = \sum_j \vec{S}_j \cdot \vec{S}_{j+1}$$

S-matrix of O(3) σ-model

- Integrable for $\vartheta=0, \pi$ [Zamolodchikov[2]]

$$\vartheta = 0 : \text{massive triplet } (+,0,-) \; m \sim e^{-\pi/g}$$

$$S(\theta) = \frac{\theta + 2\pi i}{\theta - 2\pi i}\mathbf{P}_0 + \frac{\theta + 2\pi i}{\theta - 2\pi i}\frac{\theta - \pi i}{\theta + \pi i}\mathbf{P}_1 + \frac{\theta - \pi i}{\theta + \pi i}\mathbf{P}_2$$

$$\vartheta = \pi : \text{massless L}-, \text{R}-\text{doublet}$$

$$S_{LL}(\theta) = S_{RR}(\theta) = S_{LR}(\theta) = \frac{\Gamma\left(\frac{1}{2} + \frac{\theta}{2\pi i}\right)\Gamma\left(-\frac{\theta}{2\pi i}\right)}{\Gamma\left(\frac{1}{2} - \frac{\theta}{2\pi i}\right)\Gamma\left(\frac{\theta}{2\pi i}\right)}\frac{\theta\mathbf{1} - i\pi\mathsf{P}}{\theta - i\pi}$$

- $\vartheta=\pi$: RG flow to IR CFT = $su(2)_1$ WZW

Sausage model: defined by S-matrix $SST^{(\pm)}_{\lambda}$

$SST^{(+)}_{\lambda}$: massive triplet $(+,0,-)$ $m \sim e^{-\pi/g}$

0<λ<1/2: repulsive

(λ>1/2: very complicated)

$$S^{++}_{++}(\theta) = S^{+-}_{+-}(i\pi - \theta) = \frac{\sinh(\lambda(\theta - i\pi))}{\sinh(\lambda(\theta + i\pi))},$$

$$S^{0+}_{+0}(\theta) = S^{00}_{+-}(i\pi - \theta) = \frac{-i\sin(2\pi\lambda)}{\sinh(\lambda(\theta - 2i\pi))} S^{++}_{++}(\theta),$$

$$S^{+0}_{+0}(\theta) = \frac{\sinh(\lambda\theta)}{\sinh(\lambda(\theta - 2i\pi))} S^{++}_{++}(\theta),$$

$$S^{-+}_{+-}(\theta) = -\frac{\sin(\pi\lambda)\sin(2\pi\lambda)}{\sinh(\lambda(\theta - 2i\pi))\sinh(\lambda(\theta + i\pi))}, \quad S^{00}_{00}(\theta) = S^{+0}_{+0}(\theta) + S^{+-}_{-+}(\theta)$$

Quantum group deformation of O(3) S-matrices

$SST^{(-)}_{\lambda}$: massless doublet

$$U^{++}_{++}(\theta) = U^{--}_{--}(\theta) = U_0(\theta),$$

$$U^{+-}_{+-}(\theta) = U^{-+}_{-+}(\theta) = -\frac{\sinh(\lambda\theta/(1-\lambda))}{\sinh(\lambda(\theta - i\pi)/(1-\lambda))} U_0(\theta),$$

$$U^{+-}_{-+}(\theta) = U^{-+}_{+-}(\theta) = -i\frac{\sin(\pi\lambda/(1-\lambda))}{\sinh(\lambda(\theta - i\pi)/(1-\lambda))} U_0(\theta),$$

$$U_0(\theta) = -\exp\left[i\int_0^\infty \frac{\sinh((1-2\lambda)\pi\omega/(2\lambda))\sin(\omega\theta)}{\cosh(\pi\omega/2)\sinh((1-\lambda)\pi\omega/(2\lambda))}\frac{d\omega}{\omega}\right]$$

Effective Lagrangian of Sausage model

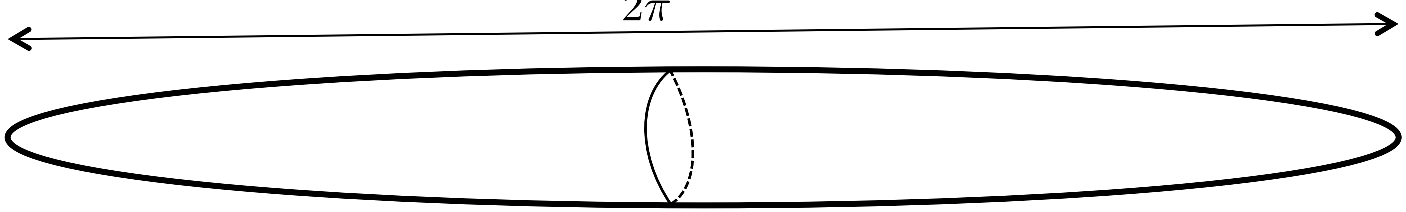
RG analysis for near UV limit (RG time t → -∞)

$$\mathcal{A}[\mathrm{SSM}_{\nu}^{(\vartheta)}] = \frac{1}{2g(t)} \int d^2x \, \frac{(\partial_\mu \vec{n})^2}{1 - \frac{\nu^2 n_3^2}{2g(t)^2}} + i\vartheta T$$

$$g(t) = \frac{\nu}{2} \coth \frac{\nu(t_0 - t)}{4\pi}$$

deformation parameter

$$L \approx \frac{\sqrt{2\nu}}{2\pi}\,(t_0 - t)$$



$$\ell \approx 2\pi\sqrt{\frac{2}{\nu}}$$

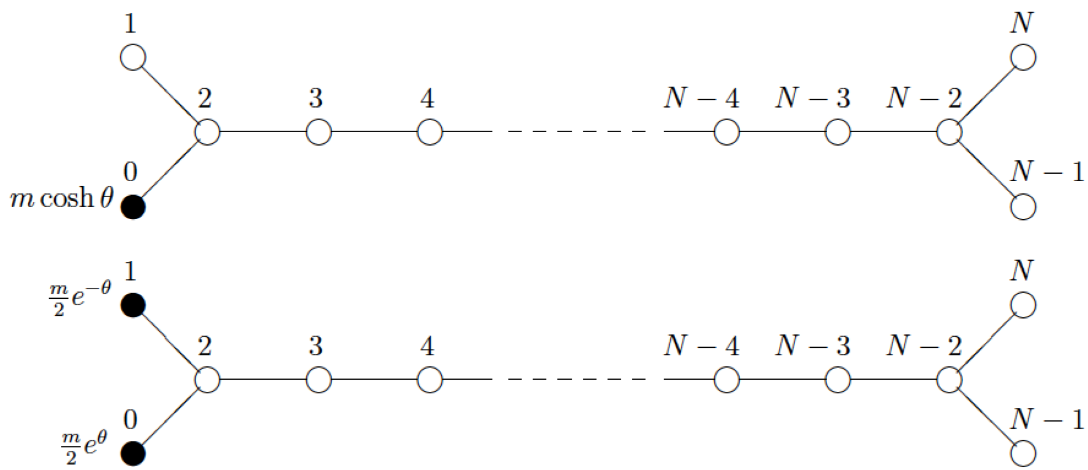
- Classical integrability: Bazhanov, Kotousov, Lukyanov 1706.09941

TBA and Y-system of $SST^{(\pm)}_{\lambda}$

- Derived only for $\lambda = \dfrac{1}{N}, \quad N = 2, 3, \ldots$

$$\epsilon_n(u) = Le_n(u) - \sum_{m=1}^{M} \int \frac{du'}{2\pi} K_{nm}(u', u) \ln\left[1 + e^{-\epsilon_m(u')}\right]$$

$$\epsilon_a(\theta) = Le_a(\theta) - \sum_{b=0}^{N} \phi_{ab} \star \ln(1 + e^{-\epsilon_b})(\theta) \qquad \phi_{ab}(\theta - \theta') = \frac{\ell_{ab}}{\cosh(\theta - \theta')}$$

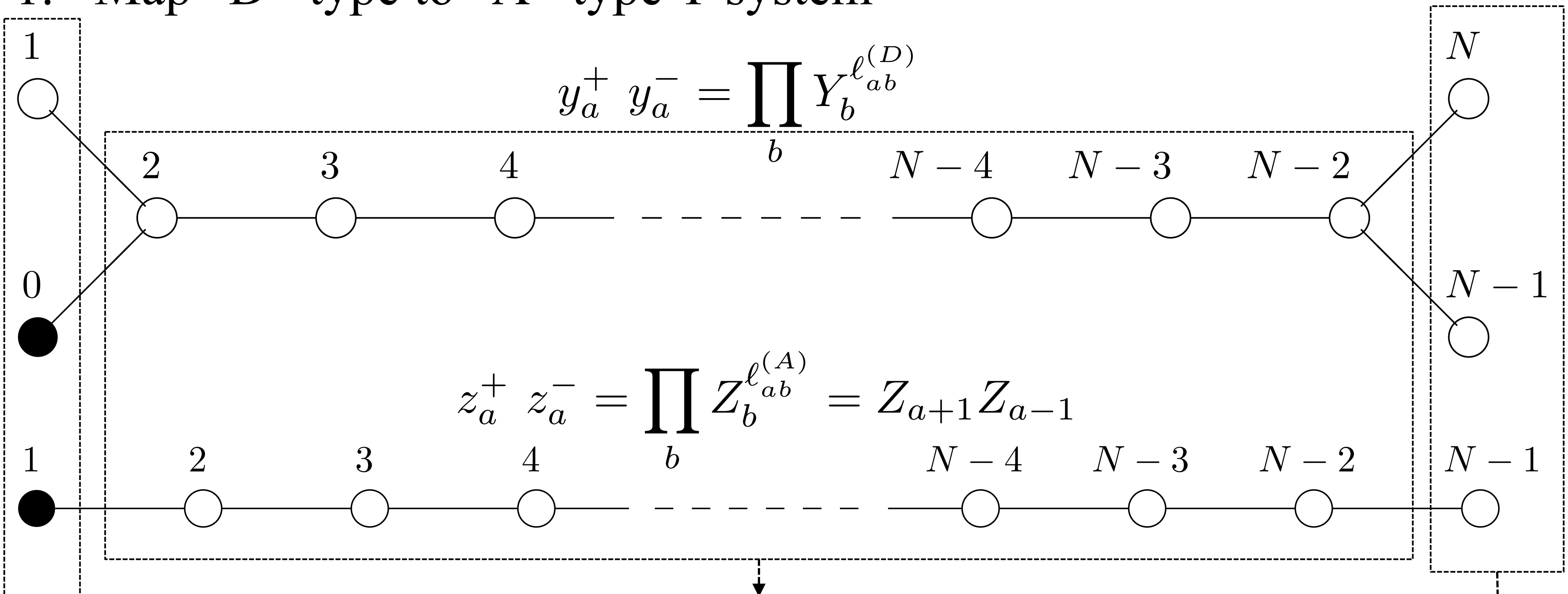


- Y-system ("D"-type)

$$y_a^+ \, y_a^- = \prod_b Y_b^{\ell_{ab}^{(D)}}, \qquad y^{\pm} = y(\theta \pm \frac{i\pi}{2})$$
$$Y = 1 + y$$

Derivation of NLIE for any λ

1. Map "D"-type to "A"-type Y-system



$$y_a^+ \; y_a^- = \prod_b Y_b^{\ell_{ab}^{(D)}}$$

$$z_a^+ \; z_a^- = \prod_b Z_b^{\ell_{ab}^{(A)}} = Z_{a+1} Z_{a-1}$$

$$z_k^+ z_k^- \;=\; Z_{k-1} Z_{k+1}, \qquad z_k \equiv y_k, \qquad k = 2, \ldots, N-2,$$

$$Z_1 \;=\; Y_0 Y_1, \qquad Z_{N-1} = Y_{N-1} Y_N$$

2. T-system

$$T_k^+ T_k^- = 1 + T_{k-1} T_{k+1}, \qquad k = 2, \ldots, N-2$$

$$z_k = T_{k-1} T_{k+1}, \qquad T_k^+ T_k^- = Z_k, \qquad k = 1, \ldots, N-1$$

with extra condition

$$Z_{N-2} = T_{N-2}^+ T_{N-2}^- = Y_{N-2} = y_N^+ y_N^- = y_{N-1}^+ y_{N-1}^- \quad \rightarrow \quad y_N = y_{N-1} = T_{N-2}$$

$$Z_{N-1} = T_{N-1}^+ T_{N-1}^- = Y_{N-1} Y_N = (1 + T_{N-2})^2 = 1 + T_{N-2} T_N \quad \rightarrow \quad \boxed{T_N = 2 + T_{N-2}}$$

3. T-Q system

$$f^{[k]}(\theta) \equiv f(\theta + \frac{i\pi k}{2})$$

$$T_{k+1} Q^{[k]} - T_k^- Q^{[k+2]} = \bar{Q}^{[-k-2]}, \qquad T_k^- \bar{Q}^{[-k]} - T_{k-1} \bar{Q}^{[-k-2]} = Q^{[k]}$$

with $\boxed{\bar{Q} = Q^{[2N]}}$

4. From T-Q to NLIE (Standard procedure for deriving NLIE)

$$b_k = \frac{Q^{[k+2]}T_k^-}{\bar{Q}^{[-k-2]}}, \qquad B_k = 1 + b_k = \frac{Q^{[k]}T_{k+1}}{\bar{Q}^{[-k-2]}}$$

Fourier transform after taking logarithms $\widetilde{f^{[\alpha]}}(\omega) = p^\alpha \tilde{f}(\omega), \qquad p \equiv e^{\frac{\omega\pi}{2}}$

$$\begin{aligned}\tilde{b}_k &= p^{k+2}\tilde{Q} + p^{-1}\tilde{T}_k - p^{-k-2}\tilde{\bar{Q}}, \\ \tilde{B}_k &= p^k\tilde{Q} + \tilde{T}_{k+1} - p^{-k-2}\tilde{\bar{Q}}\end{aligned}$$

Eliminate Q using $\tilde{\bar{Q}} = p^{2N}\tilde{Q}$

$$\begin{aligned}\tilde{b} &= \tilde{K}\,(\tilde{B} - \tilde{\bar{B}}) + p^{-1}\tilde{s}\tilde{Y}_1\tilde{Y}_0, \\ \tilde{\bar{b}} &= \tilde{K}\,(\tilde{\bar{B}} - \tilde{B}) + p\tilde{s}\tilde{Y}_1\tilde{Y}_0, \\ \tilde{y} &= p\tilde{s}\tilde{B} + p^{-1}\tilde{s}\tilde{\bar{B}}\end{aligned}$$

Now, N is just a parameter

Analytically continue: $N \to 1/\lambda$

$$\tilde{s} = \frac{1}{p + p^{-1}} = \frac{1}{2\cosh\frac{\omega\pi}{2}}$$

$$\tilde{K} = \frac{\sinh\left(\frac{\omega\pi(1-3\lambda)}{2\lambda}\right)}{2\sinh\left(\frac{\omega\pi(1-2\lambda)}{2\lambda}\right)\cosh\frac{\omega\pi}{2}}$$

NLIE in rapidity space

SST$^{(+)}$

$$\log a = K\star\log(1+a) - K^{[2\alpha]}\star\log(1+\bar{a}) + s^{[\alpha-1]}\star\left[\log(1+y)+\log(1+\xi y)\right],$$
$$\log \bar{a} = K\star\log(1+\bar{a}) - K^{[-2\alpha]}\star\log(1+a) + s^{[1-\alpha]}\star\left[\log(1+y)+\log(1+\xi y)\right],$$
$$\log y = s^{[1-\alpha]}\star\log(1+a) + s^{[\alpha-1]}\star\log(1+\bar{a})$$

SST$^{(-)}$

$$\log a = K\star\log(1+a) - K^{[2\alpha]}\star\log(1+\bar{a}) + s^{[\alpha-1]}\star\left[\log(1+\xi^{+}y)+\log(1+\xi^{-}y)\right],$$
$$\log \bar{a} = K\star\log(1+\bar{a}) - K^{[-2\alpha]}\star\log(1+a) + s^{[1-\alpha]}\star\left[\log(1+\xi^{+}y)+\log(1+\xi^{-}y)\right],$$
$$\log y = s^{[1-\alpha]}\star\log(1+a) + s^{[\alpha-1]}\star\log(1+\bar{a})$$

Vacuum energy

$$E(L) = -\frac{m}{2\pi}\int_{\infty}^{\infty} d\theta\cosh\theta\ln(1+\xi y), \quad \xi = e^{-mL\cosh\theta}$$

$$E(L) = -\frac{m}{4\pi}\int_{\infty}^{\infty} d\theta\left[e^{\theta}\ln(1+\xi^{+}y)+e^{-\theta}\ln(1+\xi^{-}y)\right], \quad \xi^{\pm} = e^{-mLe^{\pm\theta}}$$

IR limit of $SST^{(+)}$

- Linearized NLIE (mL>>1, ξ<<1)

$$a = z(1 + w + \ldots), \quad y = h(1 + u + \ldots), \qquad z = 2,\ h = 3$$

$$w = \frac{2}{3} K \star w - \frac{2}{3} K^{[2\alpha]} \star \bar{w} + s^{[\alpha-1]} \star \left(3\xi + \frac{3}{4} u \right),$$

$$\bar{w} = \frac{2}{3} K \star \bar{w} - \frac{2}{3} K^{[2\alpha]} \star w + s^{[1-\alpha]} \star \left(3\xi + \frac{3}{4} u \right),$$

$$u = \frac{2}{3} s^{[1-\alpha]} \star w + \frac{2}{3} s^{[\alpha-1]} \star \bar{w}$$

- Solutions by F.T.

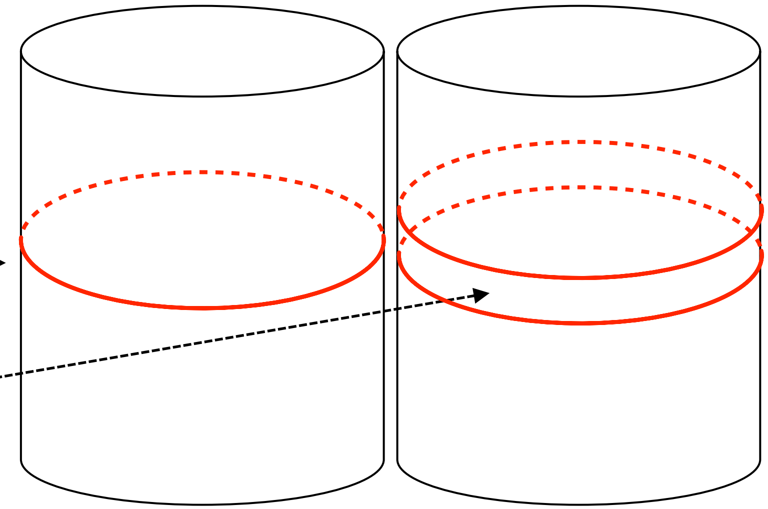
$$\tilde{u} = \frac{1}{3} \tilde{\xi} \tilde{\varphi}, \qquad \tilde{\varphi} = 8\ \frac{\sinh\left[\pi\omega\left(\frac{1}{2\lambda} - 1\right)\right]}{\sinh\frac{\pi\omega}{2\lambda}} - 4\ \frac{\sinh\left[\pi\omega\left(\frac{1}{2\lambda} - 2\right)\right]}{\sinh\frac{\pi\omega}{2\lambda}}$$

- Virial expansion

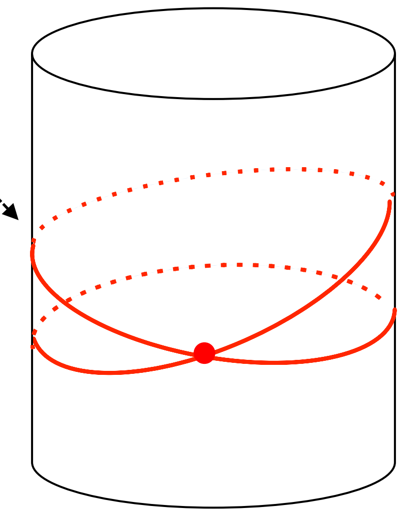
$$E = E^{(1)} + E_1^{(2)} + E_2^{(2)} + \mathcal{O}(e^{-3mL})$$

$$E^{(1)} = -\frac{e_1 m}{2\pi}\int_{\infty}^{\infty} d\theta \cosh\theta e^{-mL\cosh\theta}$$

$$E_1^{(2)} = \frac{e_2 m}{4\pi}\int_{\infty}^{\infty} d\theta \cosh\theta e^{-2mL\cosh\theta}$$



$$E_2^{(2)} = -\frac{m}{2\pi}\int_{\infty}^{\infty} d\theta \cosh\theta e^{-mL\cosh\theta}\int_{\infty}^{\infty} d\theta' \varphi(\theta-\theta') e^{-mL\cosh\theta'}$$



$$e_1 = 3, \quad e_2 = 9$$

$$\varphi(\theta) = \frac{1}{2\pi i}\frac{d}{d\theta}\log\det S_\lambda^{(+)}(\theta)$$

UV limit

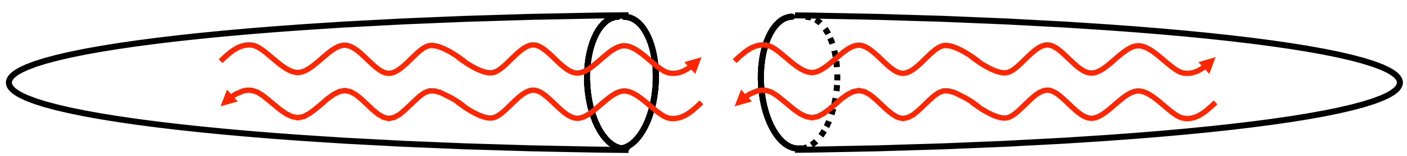
- NLIE as infinite order DE [Al.B.Zamolodchikov]

$$\delta_{a0}\, mr\cosh\theta = \varepsilon_a(\theta) + \sum_b \sum_{n=0}^{\infty} \tilde{\Psi}_{ab,n} L_b^{(n)}(\theta)$$

$$c(r) \approx 2 - \frac{3\pi^2}{2}\frac{\lambda^{-1}-2}{(\log(mr)+C)^2} + \dots$$

- Zero-mode dynamics from Lagrangian [FOZ]

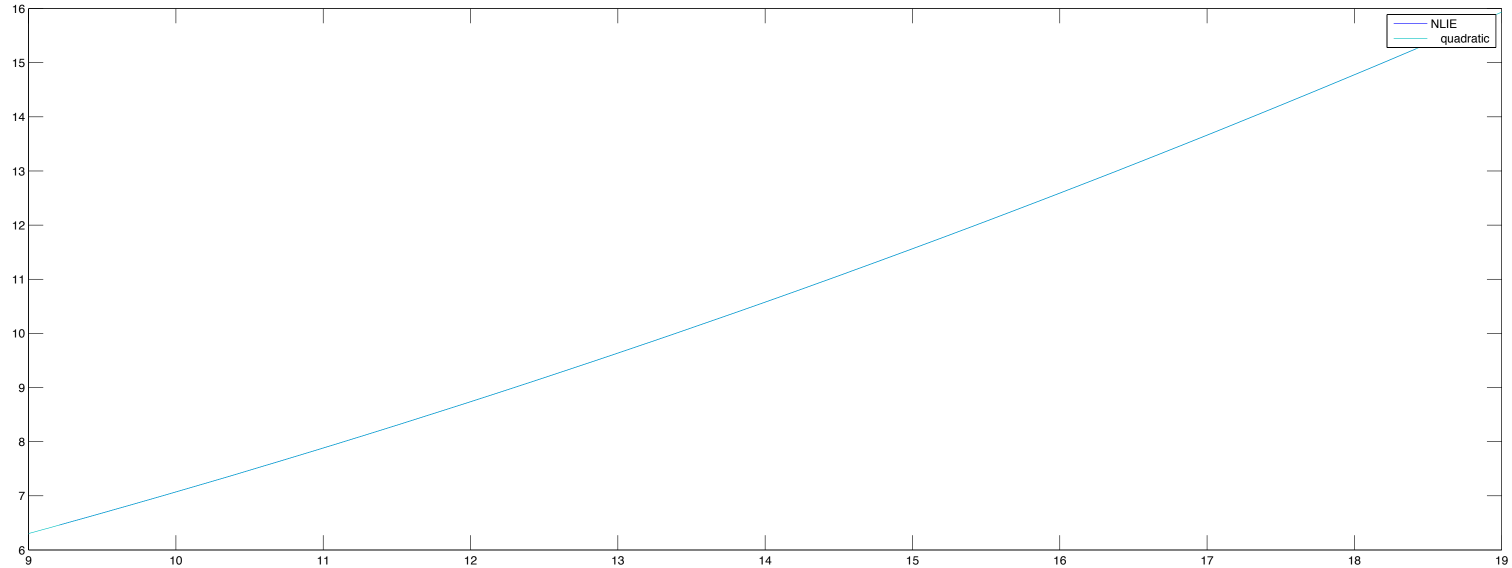
"Cigar" or sine-Liouville



$$c(r) = 2 - \frac{4\pi}{\nu}\,\frac{3\pi^2}{2(\log(mr)+\delta)^2}$$

Parametric relation: L vs. S

$$\frac{\nu}{4\pi} = \frac{\lambda}{1-2\lambda}$$



Summary and Discussion

- NLIE for any λ ($<1/2$) from TBA by analytic continuation
- Satisfy all consistent checks
- Parametric relation between ν and λ
- C. Dunning's conjecture (2002);
  Bazhanov, Kotousov, Lukyanov (1706.09941)
  propose different NLIE
  - But generates same numerics
  - Need to be clarified
- Attractive regime ($\lambda > 1/2$)?
- Applicable to other deformed NLSM?

(extra) 3. T-Q system

$$Q^{++} + Q^{--} = AQ, \qquad A = \frac{T_k^{[-k+1]} + T_{k-2}^{[-k-1]}}{T_{k-1}^{[-k]}} \qquad \bar{Q}^{++} + \bar{Q}^{--} = \bar{A}\bar{Q}, \qquad \bar{A} = \frac{T_k^{[k+3]} + T_{k+2}^{[k+1]}}{T_{k+1}^{[k+2]}}$$

with

$$A = \frac{2 + T_{N-2}^{[-N+1]} + T_{N-2}^{[-N-1]}}{T_{N-1}^{[-N]}}, \qquad \bar{A} = \frac{2 + T_{N-2}^{[N-1]} + T_{N-2}^{[N+1]}}{T_{N-1}^{[N]}}$$

$$\bar{A} = A^{[2N]} \qquad \rightarrow \qquad \bar{Q} = Q^{[2N]}$$