Review on Integrability on AdS/CFT

N=4 SYM vs. Strings on AdS_5 x S^5

(Ref) Beisert, Ahn, et.al.:

"Review of AdS/CFT Integrability: An Overview,"

Lett. Math. Phys. 99 (2012) 3.

AdS / CFT duality

- Type IIB superstrings on $AdS_5 \times S^5$

dual to

$\mathcal{N} = 4 \quad SU(N_c)$ super-Yang-Mills theory

[Maldacena (1997)]

- ‘t Hooft coupling

$$\lambda = N_c g_{\mathrm{YM}}^2$$

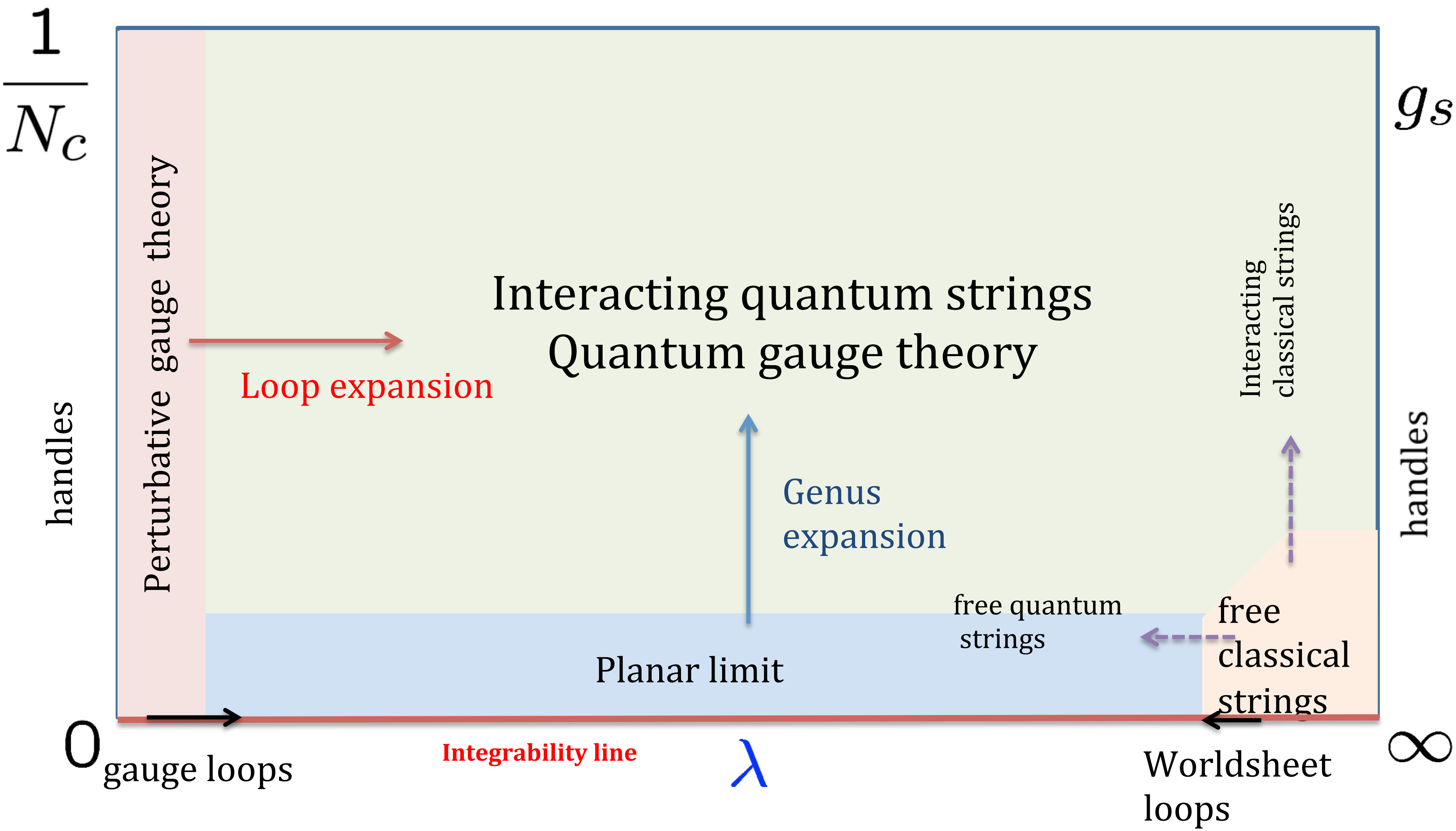
- planar limit of SYM

$$N_c \to \infty \text{ with fixed } \lambda$$

- (Related to string theory $g_s = \frac{4\pi\lambda}{N_c}$ & $\frac{R^2}{\alpha'} = \sqrt{\lambda}$)

$$g \equiv \frac{\sqrt{\lambda}}{4\pi}$$

Parameter space



Goal

Exact & complete Spectrum

For

Conformal dimension of a CFT

Energy of string configuration on AdS

$\mathcal{N}$=4 SYM

$$S = \frac{\mathrm{Tr}}{g_{\mathrm{YM}}^2} \int d^4x \left\{ -\frac{1}{4} F_{\mu\nu}^2 + (D_\mu \Phi^a)^2 + \left[\Phi^a, \Phi^b\right]^2 + \bar{\chi} D\!\!\!/ \chi - i\bar{\chi} \Gamma_a [\Phi^a, \chi] \right\}$$

$$X = \Phi_1 + i\Phi_2, \quad Y = \Phi_3 + i\Phi_4, \quad Z = \Phi_5 + i\Phi_6$$

$$V_{N=4} = |ZX - XZ|^2 + |XY - YX|^2 + |YZ - ZY|^2$$

- CFT
- Exact results based on INTEGRABILITY

Anomalous dimensions of SYM operators

$$\langle \mathcal{O}_n(x)\mathcal{O}_m(0)\rangle = \frac{\delta_{mn}}{|x|^{2\Delta_n}}$$

$$\Delta(g) = \Delta_0 + \gamma(g)$$

$$\mathcal{O}(x) = \left\{\mathrm{Tr}\left[XYZF_{\mu\nu}\chi^{\alpha}(D_{\mu}Y)\ldots\right],\ldots\right\}$$

- We considered su(2) Konishi type operators

$$\mathrm{Tr}\left[ZZXX\right], \quad \mathrm{Tr}\left[ZXZX\right]$$

Nonperturbative Integrability

Symmetry

↓

S-matrix

↙ Asymptotic Bethe ansatz

↓

Full spectrum

S-matrix

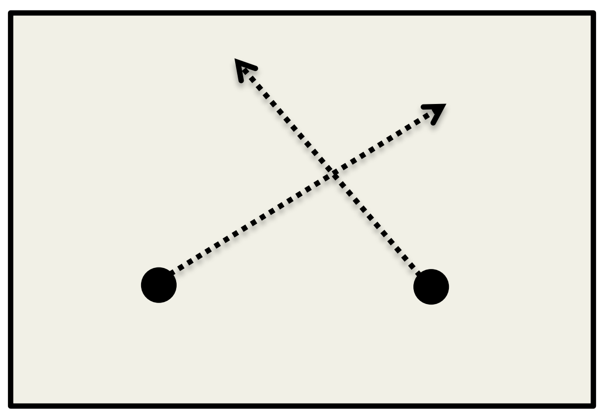
- SYM side : scattering of fields on the spin chain

$$\overrightarrow{p_1} \qquad\qquad \overleftarrow{p_2}$$

$$\mathrm{Tr}\left[\cdots ZZZZZZZZZ\chi_1 ZZZZZZ\cdots ZZZZZZ\chi_2 ZZZZZZZZZZZ\cdots\right]$$

$$\left\{X, \overline{X}, Y, \overline{Y}, \chi^{\alpha}, D_{\mu}\right\}$$

- String side : scattering on the world sheet



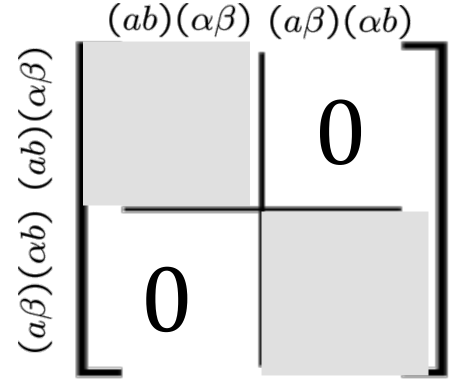
- Symmetry of the excitations: su(2|2) x su(2|2)

$$\left(\begin{array}{c|c} \mathbb{L}_a{}^b & \mathbb{Q}_\alpha{}^b \\ \hline \mathbb{Q}^{\dagger\beta}_a & \mathbb{R}_\alpha{}^\beta \end{array}\right), \quad \left(\begin{array}{c|c} \mathbb{L}_{\dot a}{}^{\dot b} & \mathbb{Q}_{\dot\alpha}{}^{\dot b} \\ \hline \mathbb{Q}^{\dagger\dot\beta}_{\dot a} & \mathbb{R}_{\dot\alpha}{}^{\dot\beta} \end{array}\right)$$

- From symmetry to S-matrix [Beisert 2008]

$$\left[\mathbf{S}(p_1,p_2), \left(\begin{array}{c|c} \mathbb{L}_a{}^b & \mathbb{Q}_\alpha{}^b \\ \hline \mathbb{Q}^{\dagger\beta}_a & \mathbb{R}_\alpha{}^\beta \end{array}\right)\right] = 0$$

- S : 16 x 16 matrix

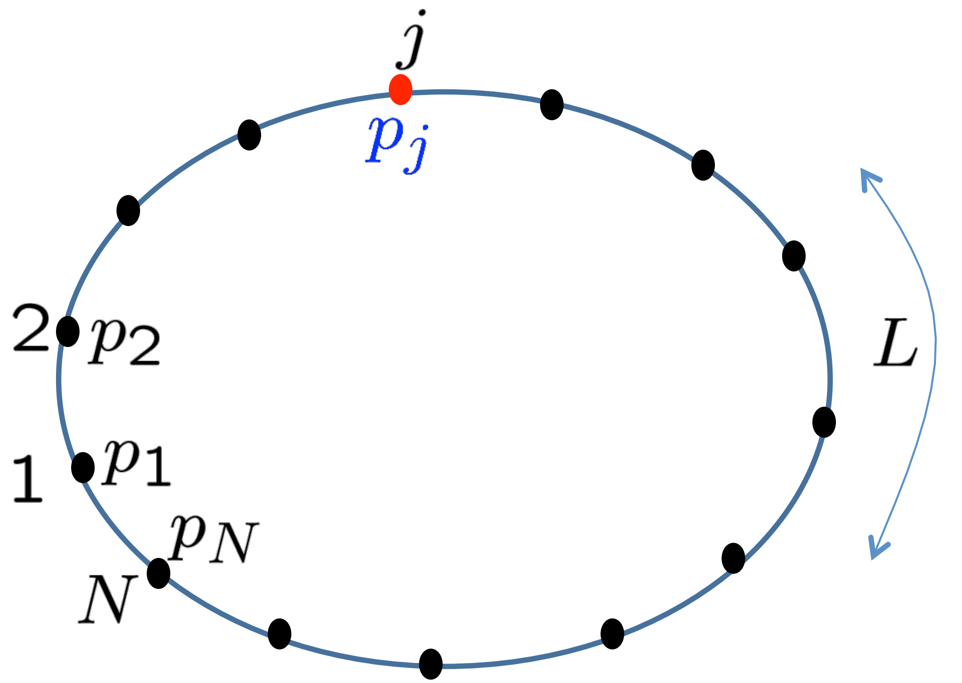


$$
\begin{aligned}
S^{aa}_{aa} &= A\,, \quad S^{\alpha\alpha}_{\alpha\alpha} = D\,,\\
S^{ab}_{ab} &= \frac{1}{2}(A-B)\,, \quad S^{ba}_{ab} = \frac{1}{2}(A+B)\,,\\
S^{\alpha\beta}_{\alpha\beta} &= \frac{1}{2}(D-E)\,, \quad S^{\beta\alpha}_{\alpha\beta} = \frac{1}{2}(D+E)\,,\\
S^{\alpha\beta}_{ab} &= -\frac{1}{2}\epsilon_{ab}\epsilon^{\alpha\beta}C\,, \quad S^{ab}_{\alpha\beta} = -\frac{1}{2}\epsilon^{ab}\epsilon_{\alpha\beta}F\,,\\
S^{a\alpha}_{a\alpha} &= G\,, \quad S^{\alpha a}_{a\alpha} = H\,, \quad S^{a\alpha}_{\alpha a} = K\,, \quad S^{\alpha a}_{\alpha a} = L
\end{aligned}
$$

$$
\begin{aligned}
A &= S_0\frac{x_2^- - x_1^+}{x_2^+ - x_1^-}\frac{\eta_1\eta_2}{\tilde\eta_1\tilde\eta_2}\,,\\
B &= -S_0\left[\frac{x_2^- - x_1^+}{x_2^+ - x_1^-} + 2\frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_2^- + x_1^+)}{(x_1^- - x_2^+)(x_1^-x_2^- - x_1^+x_2^+)}\right]\frac{\eta_1\eta_2}{\tilde\eta_1\tilde\eta_2}\,,\\
C &= S_0\frac{2ix_1^-x_2^-(x_1^+ - x_2^+)\eta_1\eta_2}{x_1^+x_2^+(x_1^- - x_2^+)(1 - x_1^-x_2^-)}\,, \qquad D = -S_0\,,\\
E &= S_0\left[1 - 2\frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^- + x_2^+)}{(x_1^- - x_2^+)(x_1^-x_2^- - x_1^+x_2^+)}\right]\,,\\
F &= S_0\frac{2i(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^+ - x_2^+)}{(x_1^- - x_2^+)(1 - x_1^-x_2^-)\tilde\eta_1\tilde\eta_2}\,,\\
G &= S_0\frac{(x_2^- - x_1^-)}{(x_2^+ - x_1^-)}\frac{\eta_1}{\tilde\eta_1}\,, \qquad H = S_0\frac{(x_2^+ - x_2^-)}{(x_1^- - x_2^+)}\frac{\eta_1}{\tilde\eta_2}\,,\\
K &= S_0\frac{(x_1^+ - x_1^-)}{(x_1^- - x_2^+)}\frac{\eta_2}{\tilde\eta_1}\,, \qquad L = S_0\frac{(x_1^+ - x_2^+)}{(x_1^- - x_2^+)}\frac{\eta_2}{\tilde\eta_2}
\end{aligned}
$$

$$\eta_1 = \eta(p_1)e^{ip_2/2}\,, \quad \eta_2 = \eta(p_2)\,, \quad \tilde\eta_1 = \eta(p_1)\,, \quad \tilde\eta_2 = \eta(p_2)e^{ip_1/2}$$

- Asymptotic Bethe ansatz
  - PBC



  - At each crossing, S-matrix $$e^{ip_jL}\prod_{k\neq j,1}^{N} S(p_j,p_k)=1$$
  - Diagonalize "transfer" matrix

$$
\begin{aligned}
1 &= \prod_{k=1}^{K_2} \frac{u_{1j}-u_{2k}+\frac{i}{2}}{u_{1j}-u_{2k}-\frac{i}{2}} \prod_{k=1}^{K_4} \frac{1-1/x_{1j}x_{4k}^{+}}{1-1/x_{1j}x_{4k}^{-}} \\
1 &= \prod_{k=1}^{K_2} \frac{u_{2j}-u_{2k}-i}{u_{2j}-u_{2k}+i} \prod_{k=1}^{K_3} \frac{u_{2j}-u_{3k}+\frac{i}{2}}{u_{2j}-u_{3k}-\frac{i}{2}} \prod_{k=1}^{K_1} \frac{u_{2j}-u_{1k}+\frac{i}{2}}{u_{2j}-u_{1k}-\frac{i}{2}} \\
1 &= \prod_{k=1}^{K_2} \frac{u_{3j}-u_{2k}+\frac{i}{2}}{u_{3j}-u_{2k}-\frac{i}{2}} \prod_{k=1}^{K_4} \frac{x_{3j}-x_{4k}^{+}}{x_{3j}-x_{4k}^{-}} \\
\left(\frac{x_{4j}^{+}}{x_{4j}^{-}}\right)^{L} &= \prod_{k=1}^{K_4} \sigma^2(x_{4j},x_{4k}) \frac{u_{4j}-u_{4k}+i}{u_{4j}-u_{4k}-i} \\
&\times \prod_{k=1}^{K_1} \frac{1-1/x_{4j}^{-}x_{1k}}{1-1/x_{4j}^{+}x_{1k}} \prod_{k=1}^{K_3} \frac{x_{4j}^{-}-x_{3k}}{x_{4j}^{+}-x_{3k}} \prod_{k=1}^{K_5} \frac{x_{4j}^{-}-x_{5k}}{x_{4j}^{+}-x_{5k}} \prod_{k=1}^{K_7} \frac{1-1/x_{4j}^{-}x_{7k}}{1-1/x_{4j}^{+}x_{7k}} \\
1 &= \prod_{k=1}^{K_6} \frac{u_{5j}-u_{6k}+\frac{i}{2}}{u_{5j}-u_{6k}-\frac{i}{2}} \prod_{k=1}^{K_4} \frac{x_{5j}-x_{4k}^{+}}{x_{5j}-x_{4k}^{-}} \\
1 &= \prod_{k=1}^{K_6} \frac{u_{6j}-u_{6k}-i}{u_{6j}-u_{6k}+i} \prod_{k=1}^{K_5} \frac{u_{6j}-u_{5k}+\frac{i}{2}}{u_{6j}-u_{5k}-\frac{i}{2}} \prod_{k=1}^{K_7} \frac{u_{6j}-u_{7k}+\frac{i}{2}}{u_{6j}-u_{7k}-\frac{i}{2}} \\
1 &= \prod_{k=1}^{K_6} \frac{u_{7j}-u_{6k}+\frac{i}{2}}{u_{7j}-u_{6k}-\frac{i}{2}} \prod_{k=1}^{K_4} \frac{1-1/x_{7j}x_{4k}^{+}}{1-1/x_{7j}x_{4k}^{-}}
\end{aligned}
$$

- Zhukovsky variables

$$x + \frac{1}{x} = u, \qquad x^{\pm} + \frac{1}{x^{\pm}} = u \pm \frac{i}{2g}$$

$$x^{+} + \frac{1}{x^{+}} - x^{-} - \frac{1}{x^{-}} = \frac{i}{g}$$

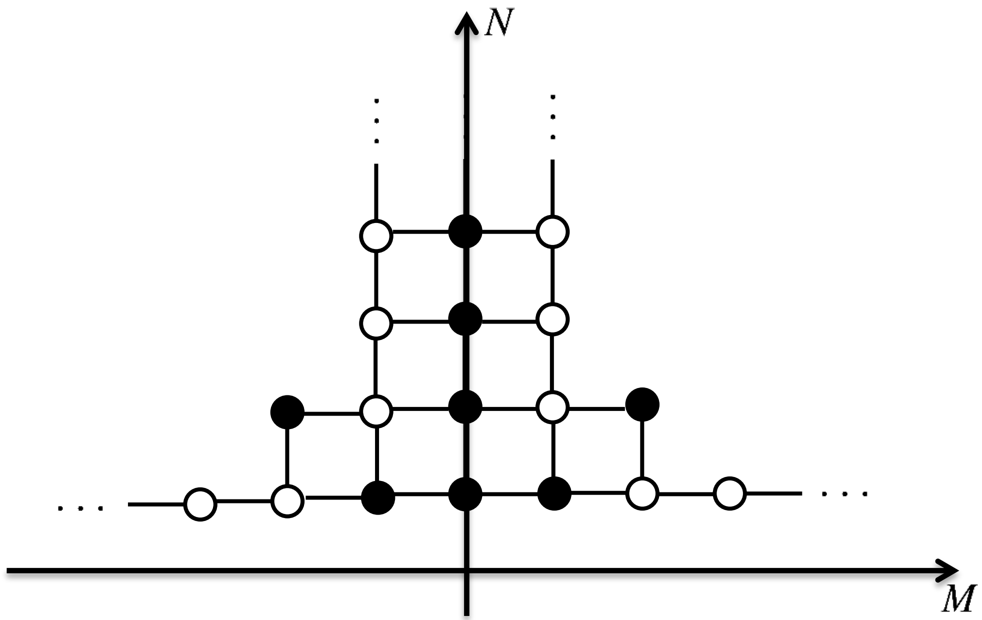
- Energy and momentum

$$\frac{x^{+}}{x^{-}} = e^{ip}, \qquad E = \frac{g}{i}\left(x^{+} - \frac{1}{x^{+}} - x^{-} + \frac{1}{x^{-}}\right)$$

$$E = \sqrt{1 + 16g^2 \sin^2 \frac{p}{2}}$$

TBA

$$\ln Y_{N,M} = s\star\Big[\ln(1+Y_{N,M+1}) + \ln(1+Y_{N,M-1})\Big] - s\star\Big[\ln(1+Y^{-1}_{N+1,M}) + \ln(1+Y^{-1}_{N-1,M})\Big]$$



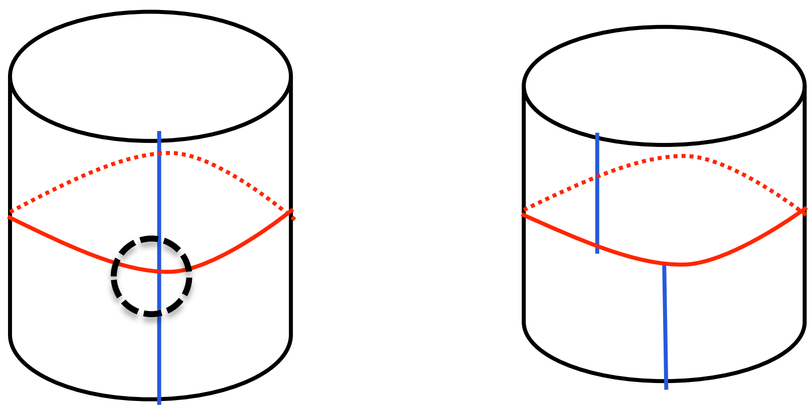
$$\Delta = -\sum_{N=1}^{\infty}\int \frac{dp}{2\pi}\log(1+Y_{N,0})$$

Strong coupling finite-size effect

- Energe correction for string configurations for $J \gg g \gg 1$

$$\delta E \approx -16g \sin^3 \frac{p}{2} \; \exp\left[-\left(\frac{J}{2g\sin\frac{p}{2}} + 2\right)\right] + \dots$$

- Luscher formula simplified when S-matrix has a pole



$$\delta E \approx -i\left[1 - \frac{e'(p)}{e'(q^*)}\right] \cdot \underset{q=q^*}{\text{res}} \sum_b S_{ba}^{ba}(q,p) \cdot e^{-iLq^*}$$

AdS/CFT in d=3, 2

- CFT side is much harder
  - even perturbative computation is difficult
  - 2d CFT is not even well-defined: Sym(T^N)
- Classical integrability for string side leads to
  - All-loop Bethe ansatz conjecture without derivation
- S-matrix is needed for
  - nonperturbative integrability: complete spectrum
  - Derivation of conjectures

d=3: ABJM

- N=6 Super-Chern-Simons theory

$$S = \frac{k}{4\pi}\int d^3x\, \mathrm{tr}\Big[D_\mu Y_a^\dagger D^\mu Y^a + \frac{1}{12}Y^aY_a^\dagger Y^bY_b^\dagger Y^cY_c^\dagger + \frac{1}{12}Y^aY_b^\dagger Y^bY_c^\dagger Y^cY_a^\dagger - \frac{1}{2}Y^aY_a^\dagger Y^bY_c^\dagger Y^cY_b^\dagger + \frac{1}{3}Y^aY_b^\dagger Y^cY_a^\dagger Y^bY_c^\dagger + \text{fermions} + \text{gauge}\Big],$$

$$(N,\bar{N}):\ Y^a = (A_1, B_1^\dagger, A_2, B_2^\dagger), \Psi_\alpha^a \qquad (\bar{N},N):\ Y_a^\dagger = (B_1, A_1^\dagger, B_2, A_2^\dagger), \bar{\Psi}_a^\alpha$$

- Planar limit:

$$\frac{k}{N} = \lambda, \qquad N,\ k \to \infty$$

- Dispersion relation: $E = \sqrt{1 + 16g^2\sin^2\frac{p}{2}}$

- But we do not know $g = h(\lambda)$ (cf) SYM: $g = \frac{\sqrt{\lambda}}{4\pi}$

- BPS vacuum with symmetry su(2|2) + su(2|2) $\left(\begin{array}{c|c} \mathbb{L}_a{}^b & \mathbb{Q}_\alpha{}^b \\ \hline \mathbb{Q}^{\dagger\beta}_a & \mathbb{R}_\alpha{}^\beta \end{array}\right)$

$$\mathrm{Tr}\left[(A_1B_1)^L\right] = \mathrm{Tr}\left[A_1B_1A_1B_1A_1B_1\cdots A_1B_1A_1B_1A_1B_1\right]$$

- Excitations
  - Composite operators as excitations over BPS vacuum

$$\mathrm{Tr}\left[A_1B_1\mathcal{A}B_1A_1B_1\cdots A_1B_1A_1\mathcal{B}A_1B_1\right]$$

"$\mathcal{A}$" : $A_2, B_2^\dagger, \Psi_1, \Psi_2$ "$\mathcal{B}$" : $B_2, A_2^\dagger, \Psi_1^\dagger, \Psi_2^\dagger$

  - ABJM fields: a fundamental rep of su(2|2) + su(2|2)

S-matrix

[Ahn-Nepomechie 2008]

$$\xrightarrow{\quad p_1 \quad} \qquad\qquad \xleftarrow{\quad p_2 \quad}$$

$$\sum_{n_1,n_2} e^{ip_1n_1+ip_2n_2}\,\mathrm{Tr}\left[\cdots A_1B_1A_1B_1\mathcal{A}_A B_1A_1B_1A_1B_1\cdots A_1B_1\mathcal{A}_B B_1A_1B_1\cdots\right]$$

$$\sum_{n_1,n_2} e^{ip_1n_1+ip_2n_2}\,\mathrm{Tr}\left[\cdots A_1B_1A_1B_1\mathcal{A}_A B_1A_1B_1A_1B_1\cdots A_1B_1A_1\mathcal{B}_B A_1B_1\cdots\right]$$

$$\left[S_{\mathcal{A}\mathcal{A}}(p_1,p_2),\left(\begin{array}{c|c}\mathfrak{R}^a{}_b & \mathfrak{Q}^\alpha{}_b\\ \hline \mathfrak{S}^a{}_\beta & \mathfrak{L}^\alpha{}_\beta\end{array}\right)\right]=\left[S_{\mathcal{A}\mathcal{B}}(p_1,p_2),\left(\begin{array}{c|c}\mathfrak{R}^a{}_b & \mathfrak{Q}^\alpha{}_b\\ \hline \mathfrak{S}^a{}_\beta & \mathfrak{L}^\alpha{}_\beta\end{array}\right)\right]=0$$

$$\begin{aligned} S_{\mathcal{A}\mathcal{A}} &= S_{\mathcal{B}\mathcal{B}} = S_0(p_1,p_2)\cdot S_{su(2|2)}(p_1,p_2)\\ S_{\mathcal{A}\mathcal{B}} &= S_{\mathcal{B}\mathcal{A}} = \tilde{S}_0(p_1,p_2)\cdot S_{su(2|2)}(p_1,p_2)\end{aligned}$$

$$S_0(p_1,p_2)=\frac{1-\frac{1}{x_1^+x_2^-}}{1-\frac{1}{x_1^-x_2^+}}\sigma_{\mathrm{BES}}(p_1,p_2),\qquad \tilde{S}_0(p_1,p_2)=\frac{x_1^- - x_2^+}{x_1^+ - x_2^-}\sigma_{\mathrm{BES}}(p_1,p_2)$$

AdS_3/CFT_2

- Strings are on $AdS_3 \times S^3 \times T^4$ / $AdS_3 \times S^3 \times S^3 \times S^1$
- Proposal for S-matrix: [Ahn-Bombardelli 2013]
- Conjecture on the excitation spectrum
  - $su(1|1) \times su(1|1)$ for $AdS_3 \times S^3 \times S^3 \times S^1$
  - $su(1|1) + su(1|1)$ for $AdS_3 \times S^3 \times T^4$

$$S(p_1,p_2) = \frac{x_1^+ - x_2^-}{x_1^- - x_2^+} \frac{1 - \frac{1}{x_1^+ x_2^-}}{1 - \frac{1}{x_1^- x_2^+}} \sigma^2(p_1,p_2) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{x_1^- - x_2^-}{x_1^+ - x_2^-} & \frac{x_1^+ - x_1^-}{x_1^+ - x_2^-} & 0 \\ 0 & \frac{x_2^+ - x_2^-}{x_1^+ - x_2^-} & \frac{x_1^+ - x_2^+}{x_1^+ - x_2^-} & 0 \\ 0 & 0 & 0 & \frac{x_1^- - x_2^+}{x_1^+ - x_2^-} \end{pmatrix}$$

- Can derive the asymptotic all-loop Bethe ansatz equation by [Babichenko-Stefanski-Zarembo]
- It seems new Zhukovsky variables needed [Tseytlin et.al.2014]

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{g}\left(1 + iqg \log \frac{x^+}{x^-}\right)$$

- Strong coupling finite-size is undergoing