Bethe asked
“What is the Bethe ansatz?”

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Hans Bethe (1906-2005)

ON THE THEORY OF METALS, I.
Eigenvalues and Eigenfunctions of a Linear Chain of Atoms

by **H. Bethe** in Rome

(Dated 13 June, 1931; received 17 June, 1931)

A method is given whereby the zero-order eigenfunctions and first-order eigenvalues (in the sense of the London-Heitler approximation scheme) are calculated for a one-dimensional "metal" consisting of a linear chain of a very large number of atoms, each of which has a single s-electron with spin, outside closed shells. In addition to the spin waves of Bloch, bound states are found, in which parallel spins are predominantly on nearest neighbor atoms; these features may be important for the theory of ferromagnetism.

Heisenberg Model

- 1D many-body Quantum Mechanics:

$$H = -J \sum_{j=1}^{N} \vec{\sigma}_j \cdot \vec{\sigma}_{j+1} = -J \sum_{j=1}^{N} \left[\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \sigma_j^z \sigma_{j+1}^z \right]$$

$$\sigma_j^a = \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \sigma^a \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} : \quad 2^N \times 2^N \text{ Matrix}$$

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- This Hamiltonian is very difficult to diagonalize
  - Numerical Method : N = ~30
  - Perturbation theory : not applicable

States

- Hilbert space: dim= 2^N $\quad |\Uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\Downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \cdots \equiv |\Uparrow\Uparrow\Downarrow\Uparrow\Downarrow\cdots\rangle, \ldots \right\}$$

- Ground State :
  - Ferromagnetic ($J > 0$) : $|\Downarrow\Downarrow\Downarrow\cdots\Downarrow\rangle$
  - Antiferromagnetic ($J < 0$) : $|\Downarrow\Uparrow\Downarrow\Uparrow\cdots\rangle + \ldots$

- Excited States :

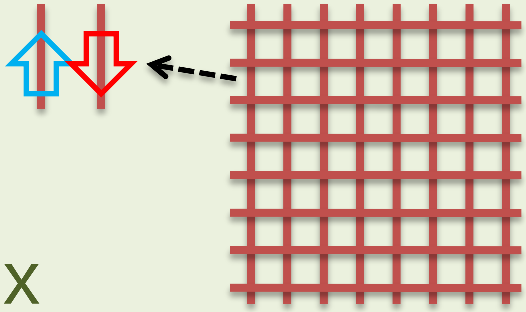
Many kinds of Bethe ansatz

- Coordinate Bethe ansatz
- Algebraic Bethe ansatz
- Functional Bethe ansatz
- Analytic Bethe ansatz
- Asymptotic Bethe ansatz
- Nested Bethe ansatz
- ….

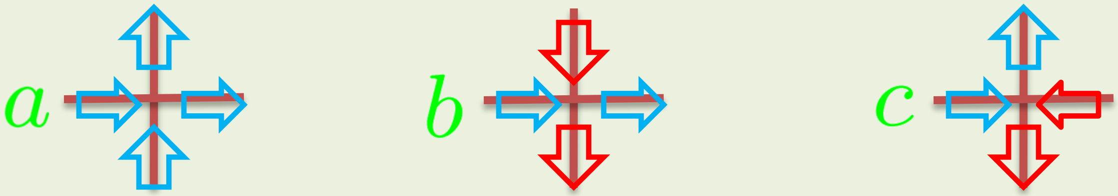
- I will concentrate on “Algebraic Bethe ansatz” since it is most general and powerful.

6 vertex model

- 2D Statistical model on square lattice



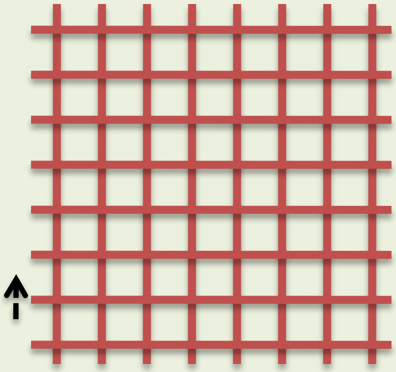
- Boltzmann Weights on each vertex



- Arrows into a vertex = arrows out of a vertex
- Symmetric under arrow inversion

6 vertex model (cont'd)

- Partition function : $Z = \sum_{\text{link}=\Uparrow,\Downarrow} \prod_{\text{vertex}} W$



- Transfer matrix :

$$Z = \mathrm{Tr}\left[T^N\right]$$

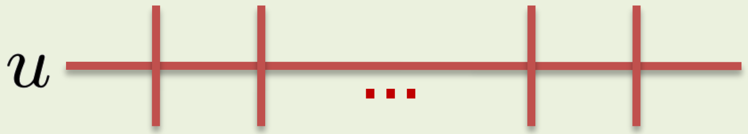
- Need to diagonalize T

6 vertex model (cont'd)

- Relation to HM:

$$T(u) = \exp\left[i\sum_{n=0} u^n Q_n\right], \quad Q_1 \equiv H_{\mathrm{HM}}$$

- Spectral parameter u is assigned on each row



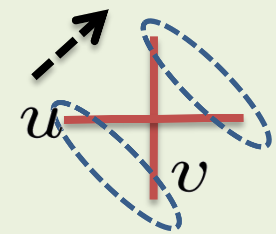
- "Integrability" : 무한개의 보존전하

$$[Q_n, Q_m] = 0 \quad \Rightarrow \quad [T(u), T(v)] = 0$$

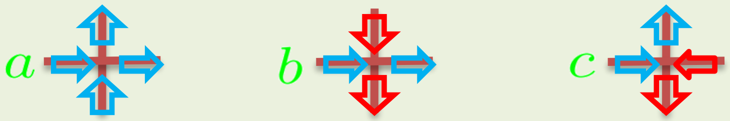
- Condition for commuting transfer matrix is …

Yang-Baxter equation

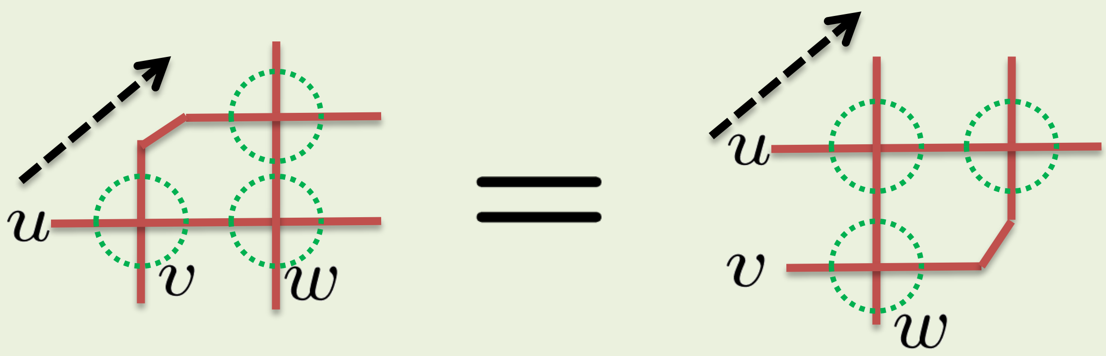
- Boltzmann weights as matrix elements :



$$W(u,v) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}(u,v)$$



- Let us assume that W satisfies YBE

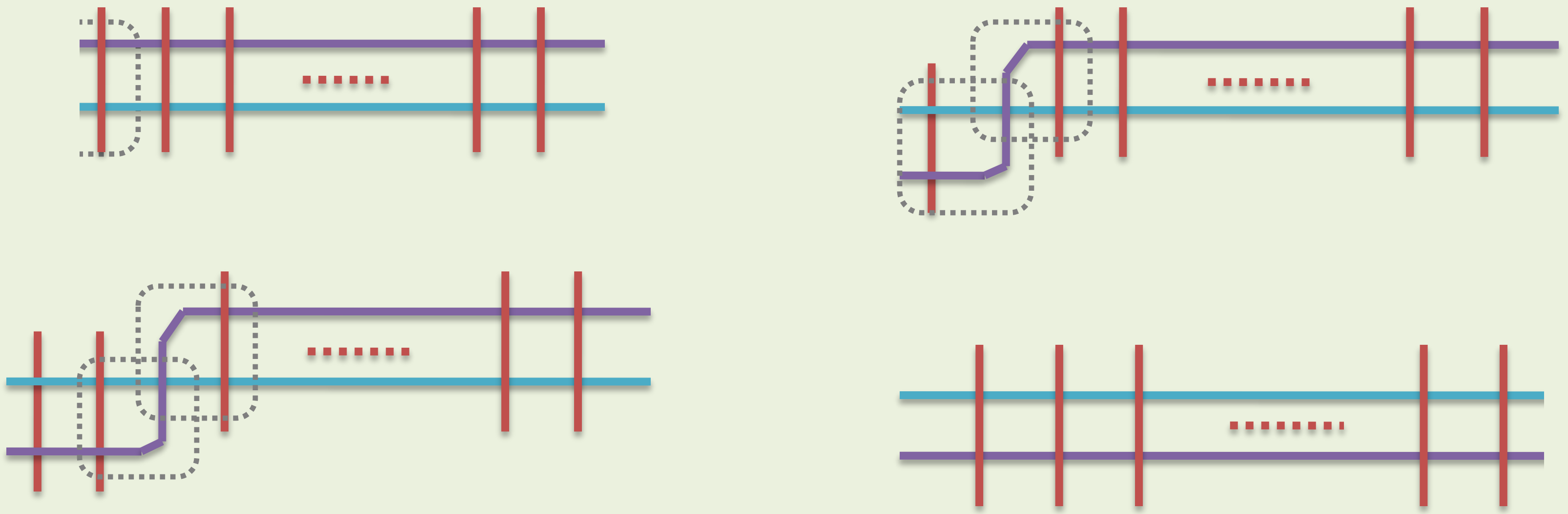


A Solution of YBE

- Solution for Heisenberg model:

$$a(u,v) = u - v + i, \;\; b(u,v) = u - v, \;\; c(u,v) = i$$

- Transfer matrices commute → Integrable



Algebraic Bethe ansatz

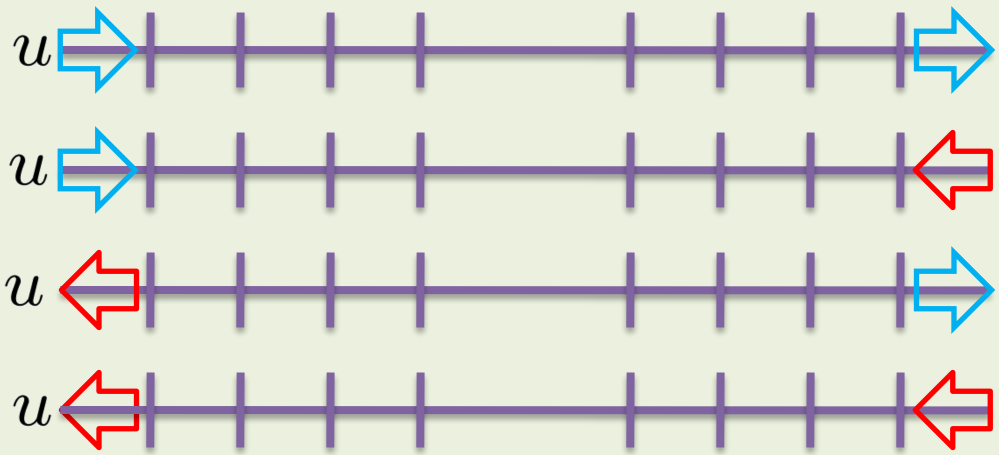
- Monodromy Matrix :

$\mathcal{A}(u)$:

$\mathcal{B}(u)$:

$\mathcal{C}(u)$:

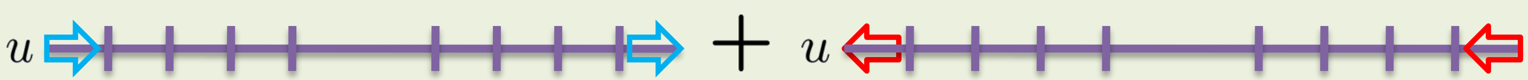
$\mathcal{D}(u)$:



- Transfer matrix for the PBC

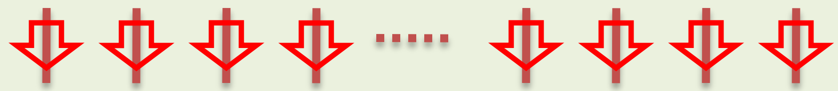
$$T(u) = \mathcal{A}(u) + \mathcal{D}(u)$$

$T(u) =$

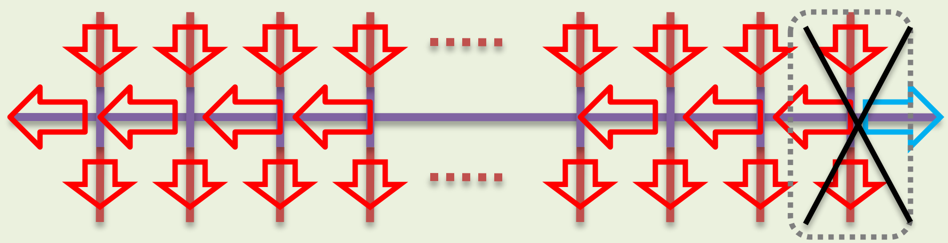


Algebraic Bethe ansatz (cont'd)

- Ferromagnetic vacuum state $|\Downarrow\Downarrow\Downarrow\cdots\Downarrow\rangle$



- Annihilation operator $\mathcal{C}(u)|\Downarrow\Downarrow\Downarrow\cdots\Downarrow\rangle = 0$

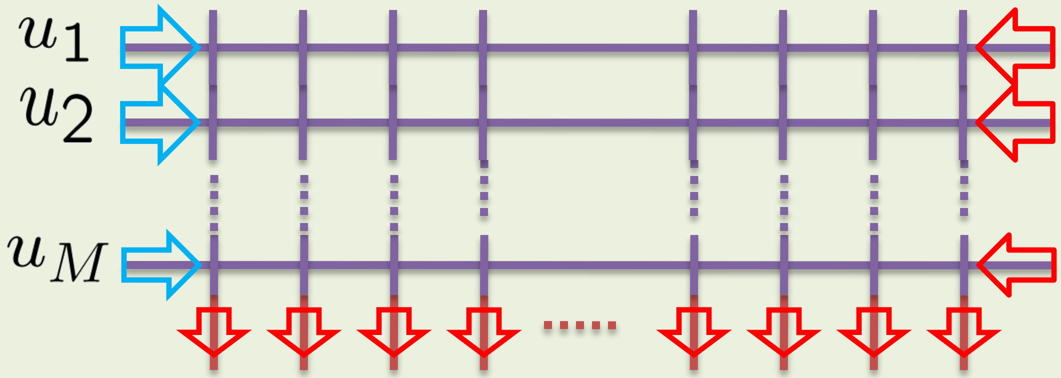


- Creation operator $\mathcal{B}(u)$

Algebraic Bethe ansatz (cont'd)

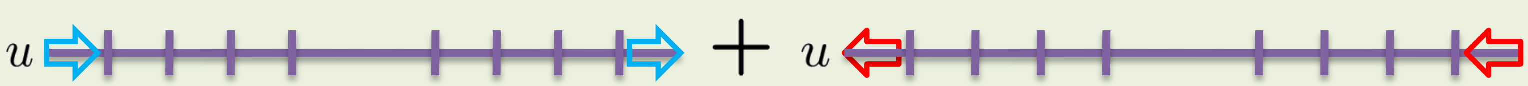
- Construct a general state

$$|\Psi(u_1,\cdots,u_M)\rangle = \mathcal{B}(u_1)\mathcal{B}(u_2)\cdots\mathcal{B}(u_M)|\Downarrow\Downarrow\Downarrow\cdots\Downarrow\rangle$$



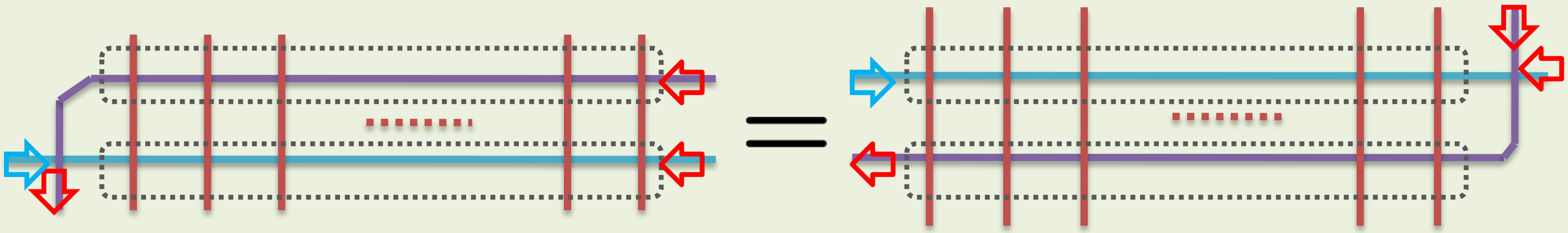
- Act the transfer matrix

$$T(u)|\Psi(u_1,\cdots,u_M)\rangle = [\mathcal{A}(u)+\mathcal{D}(u)]\mathcal{B}(u_1)\mathcal{B}(u_2)\cdots\mathcal{B}(u_M)|\Downarrow\Downarrow\Downarrow\cdots\Downarrow\rangle$$

$T(u) =$ 

YBE commutation relations

- We have seen that



$$b(u,v)\mathcal{D}(v)\mathcal{B}(u)+c(u,v)\mathcal{B}(v)\mathcal{D}(u) = a(u,v)\mathcal{B}(u)\mathcal{D}(v)$$
$$b(u,v)\mathcal{A}(v)\mathcal{B}(u)+c(u,v)\mathcal{B}(v)\mathcal{A}(u) = a(u,v)\mathcal{B}(u)\mathcal{A}(v)$$

- Act *A* & *D* on the state |Ψ> using CR

$$\mathcal{D}(u)\mathcal{B}(u_1)\mathcal{B}(u_2)\cdots\mathcal{B}(u_M)|\Downarrow\Downarrow\Downarrow\cdots\Downarrow\rangle$$

$$\mathcal{D}(u)\mathcal{B}(u_j) = \frac{a(u,v)}{b(u,v)}\mathcal{B}(u_j)\mathcal{D}(u) - \frac{c(u,v)}{b(u,v)}\mathcal{B}(u)\mathcal{D}(u_j)$$

- Many "unwanted terms" from *A* & *D* cancel each other if the Bethe ansatz equation is satisfied

Bethe ansatz Equation

- M coupled equations

$$\left(\frac{u_j+\frac{i}{2}}{u_j-\frac{i}{2}}\right)^N = \prod_{k=1}^{M} \frac{u_j-u_k+i}{u_j-u_k-i}, \qquad j=1,\ldots,M$$

- Eigenvalues of Conserved charges

$$Q_n = J\frac{i}{n}\sum_{j=1}^{M}\left[\left(u_j+\frac{i}{2}\right)^{-n}-\left(u_j-\frac{i}{2}\right)^{-n}\right], \qquad E = J\sum_{j=1}^{M}\frac{1}{{u_j}^2+\frac{1}{4}}$$

- Taking logarithm

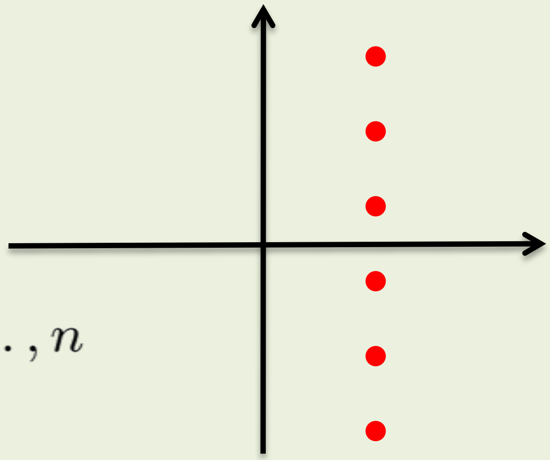
$$N\log\left(\frac{u_j+\frac{i}{2}}{u_j-\frac{i}{2}}\right)-\sum_{k=1}^{M}\log\frac{u_j-u_k+i}{u_j-u_k-i}=2\pi i I_j$$

  – Intergers I's are chosen such that total states are 2^N

Ferromagnetic vacuum

- $J > 0$: E increases along with M

- "string" solution as N →∞ :



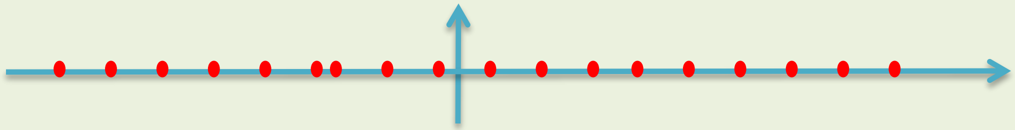
$$u_j^{(n)} = u_0 + \frac{n+1-2j}{2} i, \quad j = 1, \ldots, n$$

$$E^{(n)} = J \sum_{j=1}^{n} \frac{1}{\left(u_j^{(n)}\right)^2 + \frac{1}{4}} = \frac{n}{u_0{}^2 + \frac{n^2}{4}} \leq \frac{n}{u_0{}^2 + \frac{1}{4}} = nE^{(1)}$$

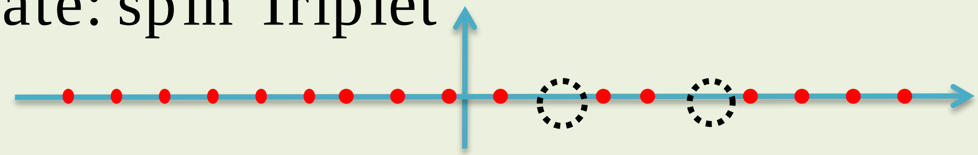
- Low lying states are given by "long strings" rather than real roots

Antiferromagnetic vacuum

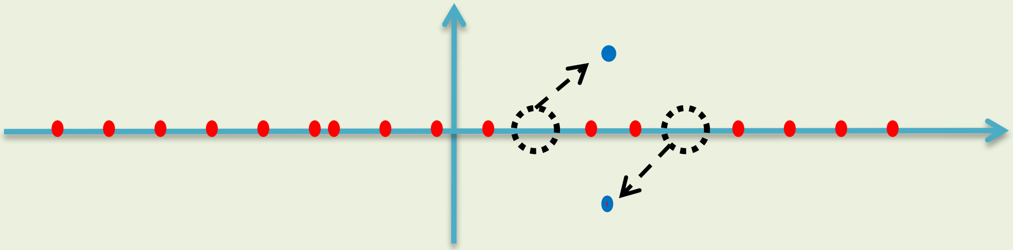
- $J < 0$: E decreases along with M
- Vacuum is given by maximum real roots $M=N/2$

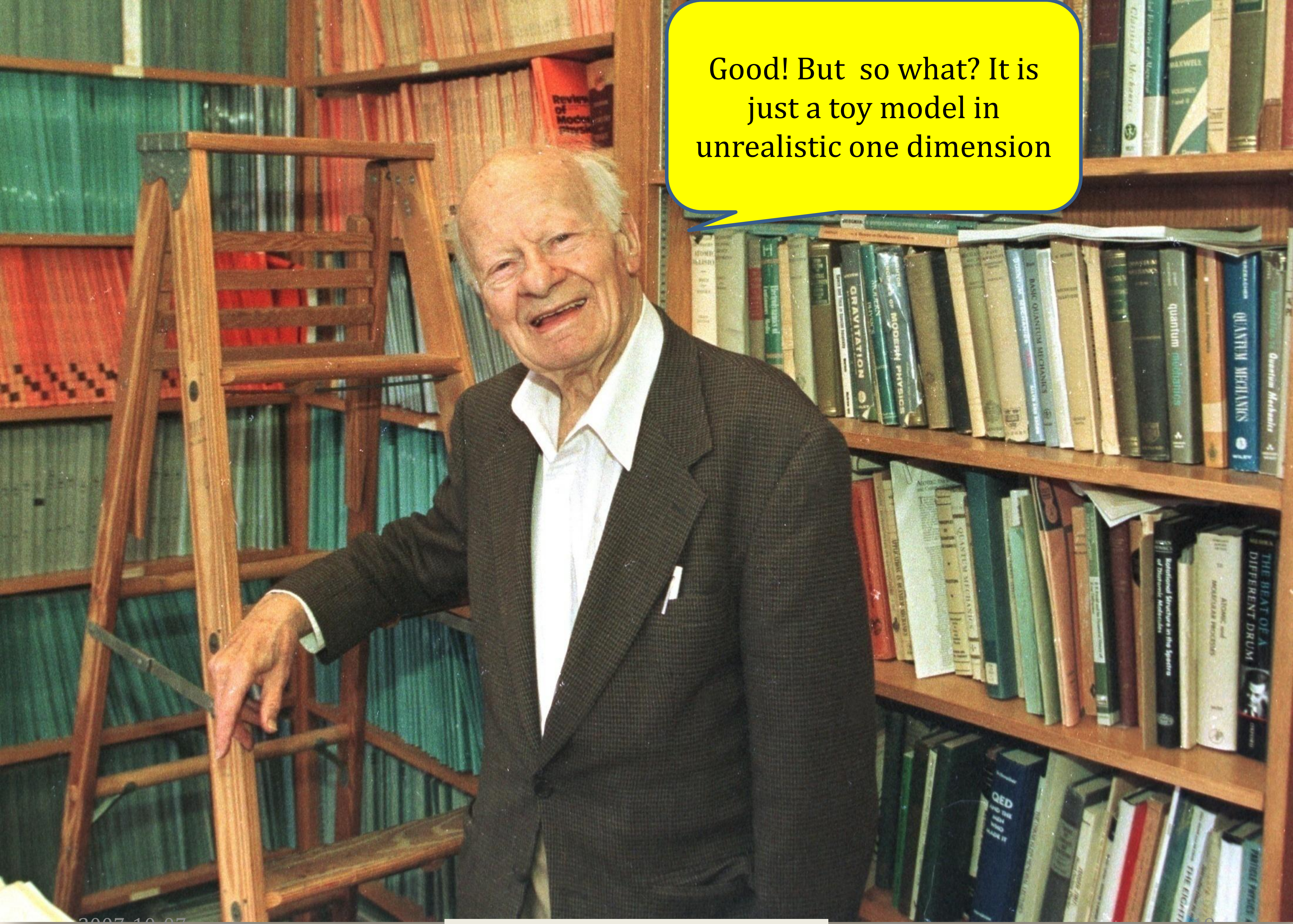


- Excited states : (ex) the first excited states
  - Two Hole (spinon) state: spin Triplet



  - Two Hole and one 2-string state: spin Singlet





Hans Bethe (1906-2005)

HM is not a toy model

- Materials described by HM exist !

PHYSICAL REVIEW B VOLUME 48, NUMBER 17 1 NOVEMBER 1993-I

Muon-spin-rotation measurements in infinite-layer and infinite-chain cuprate antiferromagnets: $Ca_{0.86}Sr_{0.14}CuO_2$ and Sr_2CuO_3

A. Keren, L. P. Le, G. M. Luke, B. J. Sternlieb,* W. D. Wu, and Y. J. Uemura
Department of Physics, Columbia University, New York, New York 10027

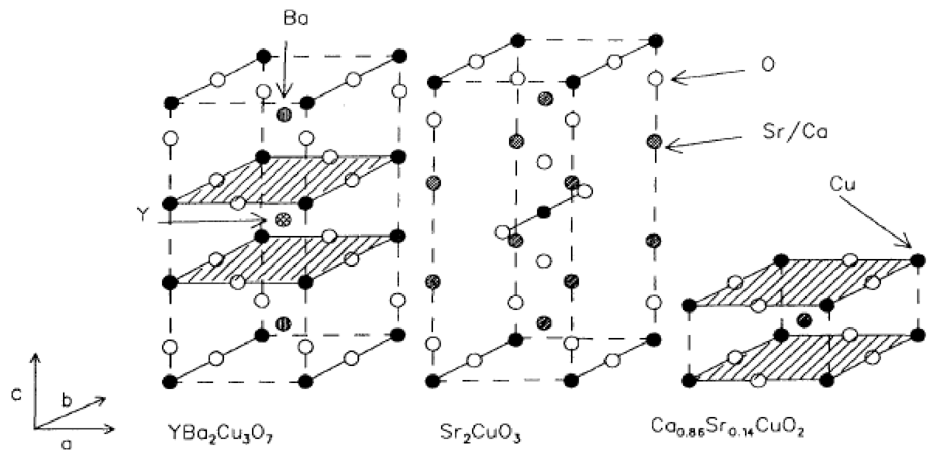
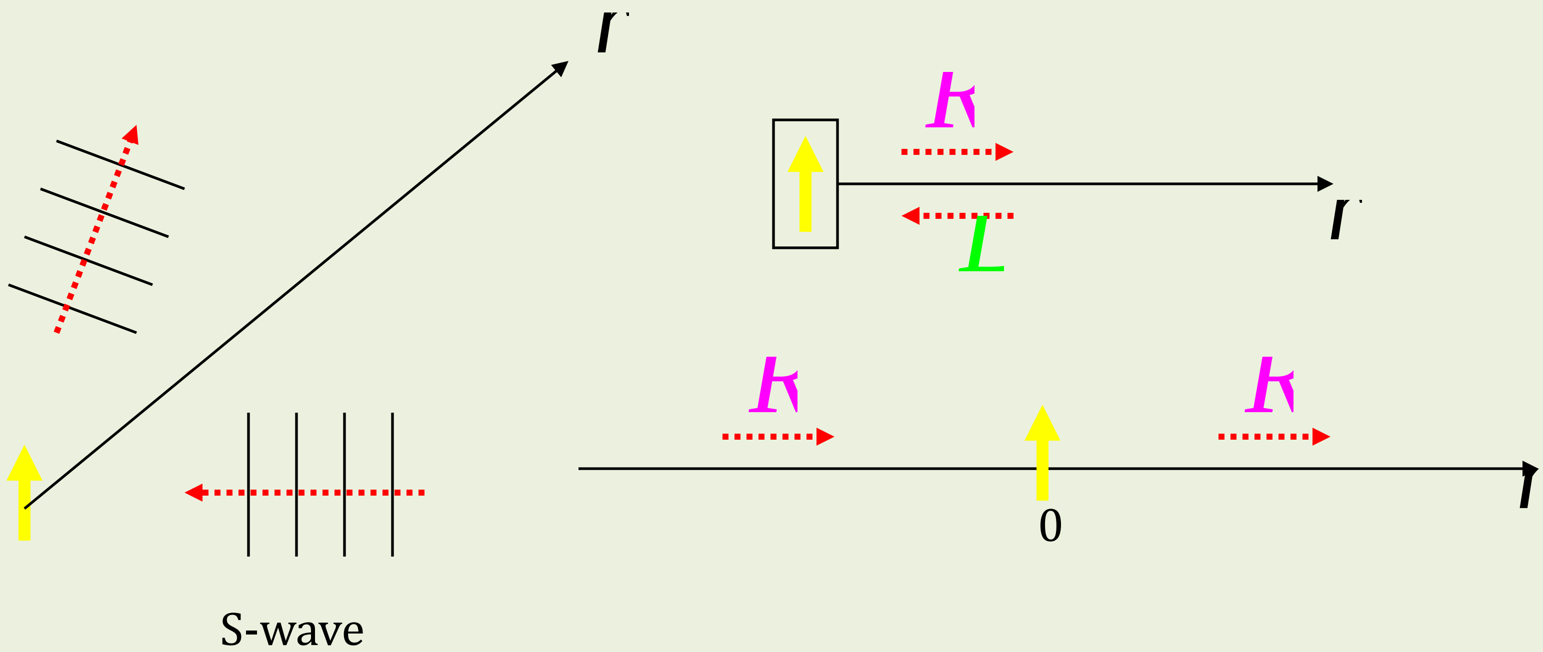


FIG. 1. Crystal structures of $YBa_2Cu_3O_7$, Sr_2CuO_3, and $Ca_{0.86}Sr_{0.14}CuO_2$. The exchange couplings along layers and chains are emphasized by the solid lines.

1D is not unrealistic

- There are many materials with effective one dimensional structure (ex) Kondo effect



Modern Application :

강한 상호작용과 초끈이론

AdS / CFT duality

- Type IIB superstrings on $AdS_5 \times S^5$ dual to $\mathcal{N} = 4$ $SU(N_c)$ Super-Yang-Mills gauge theory in 4d [Maldacena (1997)]

$\mathcal{N}$=4 super-Yang-Mills theory

- 전자기 :

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\left(\mathbf{E}^2 - \mathbf{B}^2\right) \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

- Yang-Mills theory : $A_\mu = N_c \times N_c$ Matrix

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$$

$$\mathcal{L} = -\frac{1}{4}\mathrm{Tr}\left[F_{\mu\nu}F^{\mu\nu}\right] = \frac{1}{2}\sum_a \left(\mathbf{E}_a{}^2 - \mathbf{B}_a{}^2\right)$$

- Supersymmetry : *N=4* gauge supermultiplet

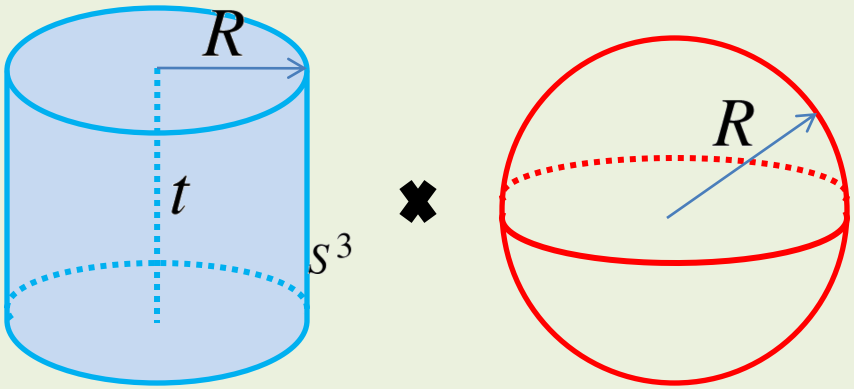
$$(A_\mu, \chi^a_\alpha, \Phi^j), \quad a = 1, \ldots, 4;\ j = 1, \ldots, 6$$

$$S = \frac{1}{g^2}\int d^4x \mathrm{Tr}\left\{\frac{1}{2}F^2_{\mu\nu} + (D_\mu \Phi^i)^2 - \frac{1}{2}([\Phi^i, \Phi^j])^2 + \ldots\right\}$$

Superstring on AdS background

- Type IIB superstrings on $AdS_5 \times S^5$ is described by a sigma model

$$S = \frac{R^2}{\alpha'} \int d\tau d\sigma \left[G_{mn}^{(\mathrm{AdS})} \partial_a X^m \partial^a X^n + G_{mn}^{(S^5)} \partial_a Y^m \partial^a Y^n + \text{fermions} \right]$$



- Full quantization is not understood

AdS / CFT duality

- Parameter relations:

$$g_s = \frac{4\pi\lambda}{N_c} \quad \& \quad \frac{R^2}{\alpha'} = \sqrt{\lambda}$$

  with 't Hooft coupling $\lambda = N_c g^2$

- Free superstring theory corresponds to a planar limit of SYM

$$g_s \to 0 \;\equiv\; N_c \to \infty \text{ with fixed } \lambda$$

- Quantitative check is tricky since it is a strong-weak duality
  – SYM perturbation for $\lambda << 1$
  – String perturbation for $\alpha' << 1 \quad \Rightarrow \quad \lambda >> 1$

Composite SYM operators

- Composite operators :

$$\mathcal{O}(x) = \mathrm{Tr}\left[\Phi^i F_{\mu\nu}\chi^\alpha (D_\mu \Phi^j)\ldots\right]$$

- Conformal dimension :

$$\langle \mathcal{O}_n(x)\mathcal{O}_m(0)\rangle = \frac{\delta_{mn}}{|x|^{2\Delta_n}}$$

can be calculated by "renormalization group"

- We will focus on a special sector : $X \equiv \Phi_1 + i\Phi_2,\ Y \equiv \Phi_3 + i\Phi_4$

$$\left\{\mathrm{Tr}\left[X^N\right], \mathrm{Tr}\left[X^{N-1}Y\right], \mathrm{Tr}\left[X^{N-n-1}YX^{n-1}Y\right], \ldots, \mathrm{Tr}\left[Y^N\right]\right\}$$

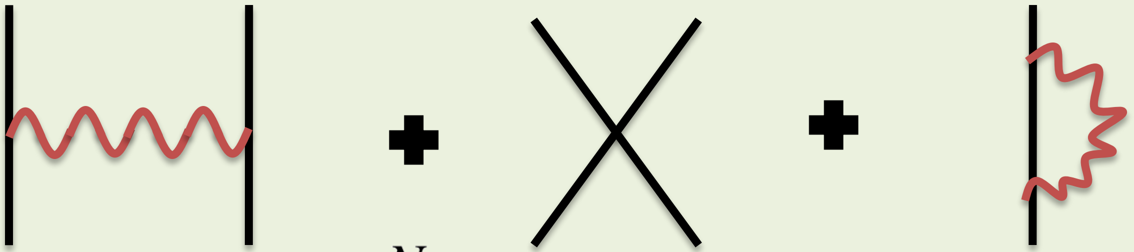
- Renormalization group mixes the composite operators $\mathcal{O}_a = Z_a^b \mathcal{O}_b$

Anomalous Dimension

- Conformal Dimension is $\Delta = N + \gamma$
- Anomalous dimension is given by a matrix

$$\Gamma = \frac{dZ}{d\log\Lambda}\cdot Z^{-1}$$

- One-Loop perturbation theory : Heisenberg model [Minahan & Zarembo]



$$\Gamma = -\frac{\lambda}{8\pi^2}\sum_{j=1}^{N}\left(\vec{\sigma}_j\cdot\vec{\sigma}_{j+1} - 1\right)$$

$Y \equiv \Uparrow, \quad X \equiv \Downarrow$

SYM Bethe ansatz

- Ferromagnetic vacuum :

$$|\Downarrow\Downarrow\Downarrow\cdots\Downarrow\rangle \equiv \mathrm{Tr}\left[X^N\right]$$

- Two "magnon" state :

$$|\Uparrow\Uparrow\Downarrow\cdots\Downarrow\rangle + \ldots \equiv \mathrm{Tr}\left[Y^2 X^{N-2} + \ldots\right]$$

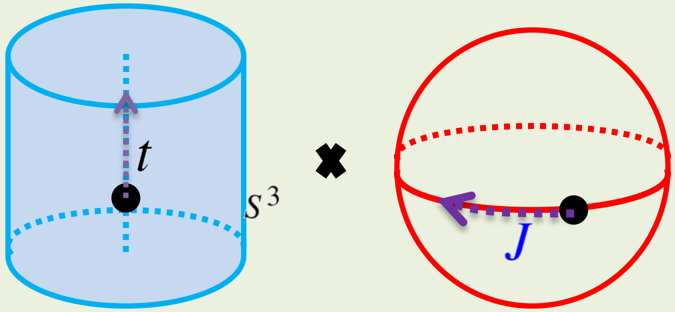
$$\left(\frac{u_1+\frac{i}{2}}{u_1-\frac{i}{2}}\right)^N = \frac{u_1-u_2+i}{u_1-u_2-i} = \frac{u_1+\frac{i}{2}}{u_1+\frac{i}{2}} \quad \text{with} \quad u_1 = -u_2$$

$$\gamma = \frac{\lambda}{\pi^2}\sin^2\frac{n\pi}{N-1} = \frac{\lambda}{\pi^2}\sin^2\frac{p}{2}$$

$$\frac{u+\frac{i}{2}}{u-\frac{i}{2}} \equiv e^{ip}$$

String theory : BMN Limit

- Point-like string moving in $AdS_5 \times S^5$ with very large angular momentum $J >> 1$



- Effective action : $$S = \sqrt{\lambda}\int d\tau d\sigma \left[\frac{1}{2}(\partial_a x^i)^2 - \frac{J^2}{2\lambda}(x^i)^2 + \text{fermions}\right]$$

- Energy : $$E - J = \sum_{n=-\infty}^{\infty}\sqrt{1+\frac{\lambda n^2}{J^2}}\hat{N}_n$$
  - Exact for all orders
  - Agrees with BAE when $\lambda << 1$

Non-perturbative SYM

- Conjecture for all-loop magnon energy

$$E = \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p}{2}}$$

$J >> 1$ → $\sqrt{1 + \frac{\lambda n^2}{J^2}}$

$\lambda << 1$ → $1 + \frac{\lambda}{2\pi^2}\sin^2\frac{p}{2}$

- Notice that higher conserved charges

$$Q_n = \frac{2^{n+1}}{n}\sin\frac{np}{2}\sin^n\frac{p}{2} \quad \Rightarrow \quad E = \sum_{n=\text{odd}} c_n Q_n$$

All-Loop Bethe ansatz

- Conjecture 1 [Beisert & Staudacher]

$$\left(\frac{x^+(u_j)}{x^-(u_j)}\right)^N = \prod_{k=1}^{M} \frac{u_j - u_k + i}{u_j - u_k - i}$$

$$x^\pm(u) = x\left(u \pm \frac{i}{2}\right) \quad \text{with} \quad x(u) \equiv \frac{1}{2}\left(u + \sqrt{u^2 - \frac{\lambda}{\pi^2}}\right)$$

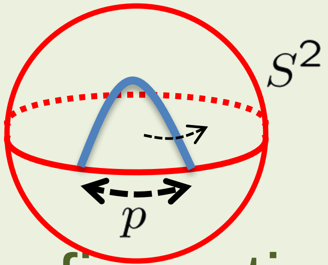
  - Matches well with perturbative theories upto 3 loops
  - Correction: Integrability and symmetry lead to
- Conjecture 2 [Beisert, Eden & Staudacher]

$$\left(\frac{x^+(u_j)}{x^-(u_j)}\right)^N = \prod_{k=1}^{M} \left[\sigma(u_j, u_k)\frac{u_j - u_k + i}{u_j - u_k - i}\right]$$

Large coupling limit-classical string limit

- In the classical string limit, the magnon energy is

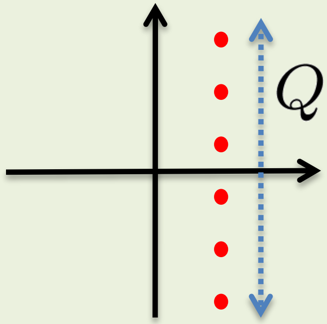
$$E = \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p}{2}} \approx \frac{\sqrt{\lambda}}{\pi}\sin\frac{p}{2}$$



and identified with a classical soliton configuration called "Giant magnon" [Hoffman&Maldacena]

- "Bethe string" → "Dyonic giant magnon"

$$E^{(Q)} = \sqrt{Q^2 + \frac{\lambda}{\pi^2}\sin^2\frac{p}{2}}$$



비섭동적 양-밀즈 /초끈 이론

- String Bethe ansatz : SU(2) sector

$$\left(\frac{x^+(u_j)}{x^-(u_j)}\right)^N = \prod_{k=1}^{M}\left[\sigma(u_j,u_k)\frac{u_j-u_k+i}{u_j-u_k-i}\right]$$

- All-loop Bethe ansatz for full sector PSU(2,2|4) are known [Beisert]

- Non-perturbative Yang-Mills theory is one of the most important problems in theoretical physics and we are moving closer to the goal !

Heisenberg Model is applicable to

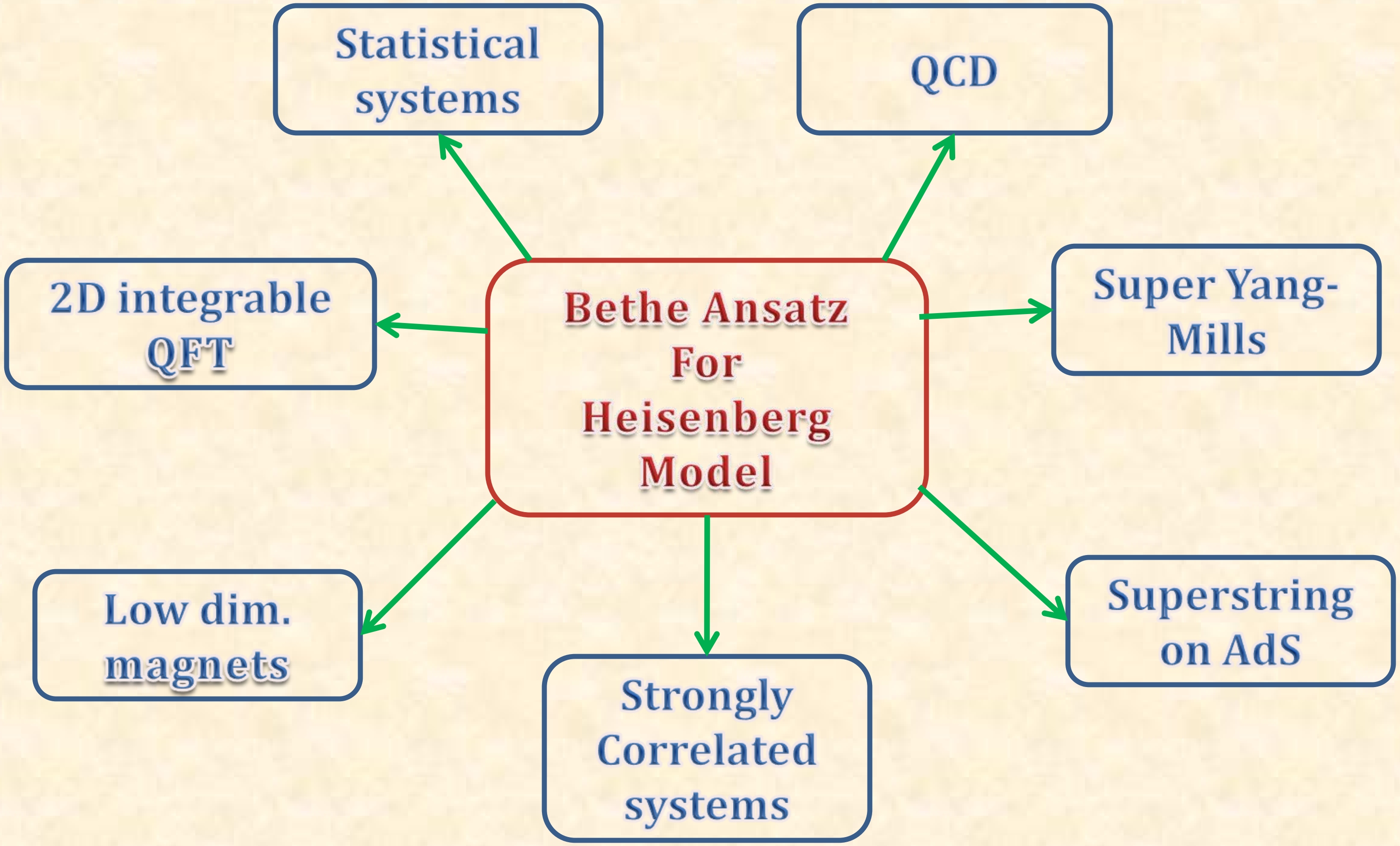
- Scale dependence of composite (Wilson) operators in QCD

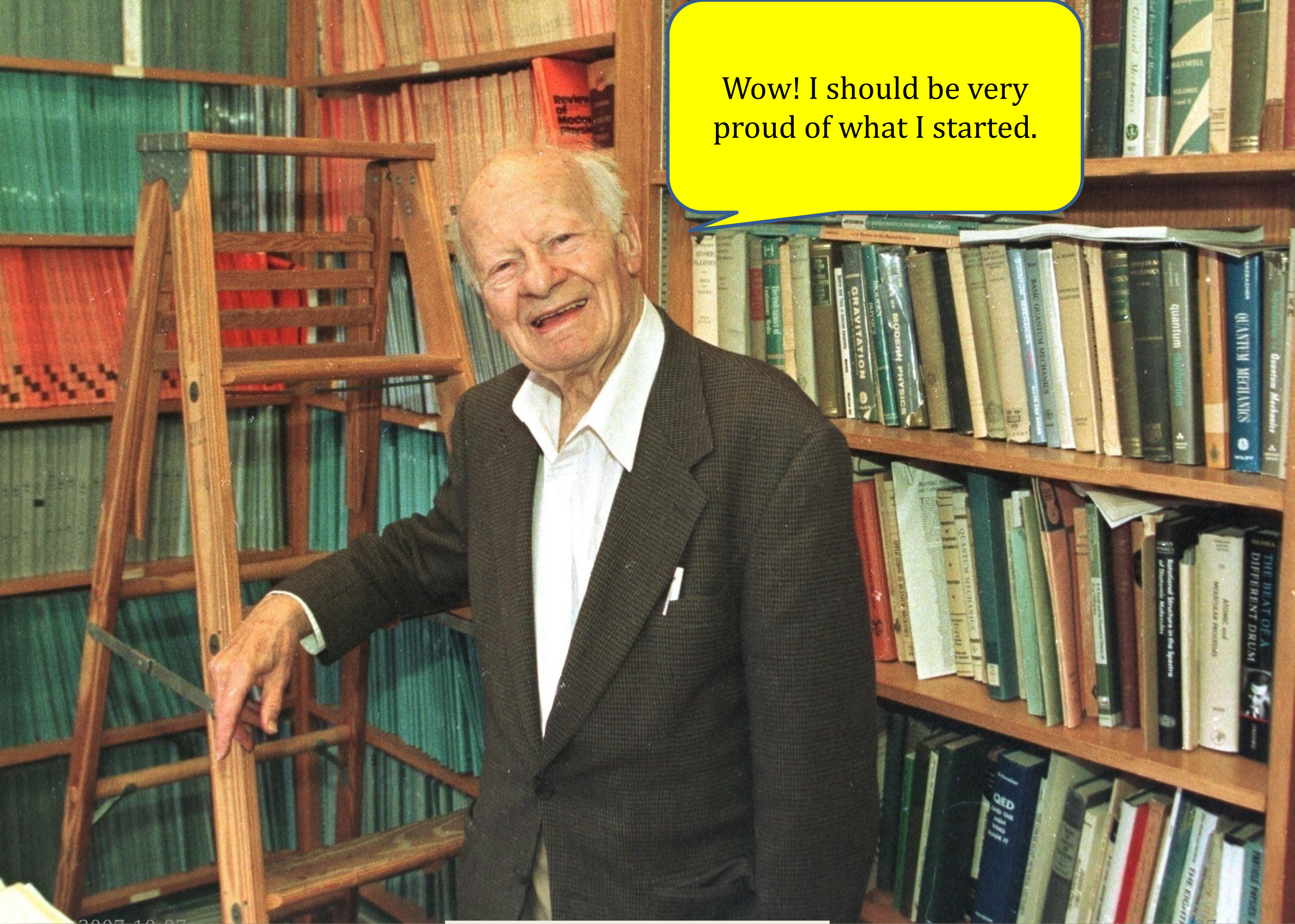
- High energy (Regge) behavior of scattering amplitudes in QCD

- Related to 2D quantum field theory like sine-Gordon model, HM can describe
  - Edge states in Fractional Quantum Hall
  - Mott insulator and transitions
  - Etc.

- Related many other “integrable lattice models”
  - XXZ (6vertex), XYZ (8 vertex), RSOS, ….

Perspective





Hans Bethe (1906-2005)