

Soliton

② $U_{xx} + U U_x + U_{xxx} = 0$

$$\frac{d^2 y}{dx^2} + \left[\underset{\uparrow}{\lambda} - U(x,t) \right] \underset{\substack{\uparrow \\ (x,\lambda,t)}}{y} = 0 \quad \Leftrightarrow \quad \textcircled{1} \quad U(x,t) = U(x+at)$$

$$L y = \lambda y \quad L = D^2 - U(x,t)$$

$$(L y)_t = (\lambda y)_t = L_t y + L y_t = y_{xxt} - (u y)_t$$

$$= L y_t - u_t y \Rightarrow L_t = -u_t$$

$$-u_t y + L y_t = \lambda_t y + \lambda y_t$$

$$y_t = B y \quad -u_t y + L B y - \underbrace{\lambda B y}_{B \lambda y = B L y} = \lambda_t y$$

$$\therefore \{ [L, B] - u_t \} y = \lambda_t y = 0$$

for $\lambda_t = 0$

$$[L, B] = [D^2 - u, B] = u_t$$

(ex) $B_1 = a D$

$$[L, B_1] y = L a D y - a D L y = D(D(a y_x)) - u a y_x - a D(y_{xx} - u y)$$

$$= D(a_x y_x + a y_{xx}) - u a y_x - a D(y_{xx} - u y)$$

$$= a_{xx} y_x + 2 a_x y_{xx} + a y_{xxx} - u a y_x - a y_{xxx} + a u_x y + a u y_x$$

$\therefore$ if $a = \text{const} \rightarrow a_x = a_{xx} = 0 \rightarrow a u_x y - u_t y = 0$

or $a u_x - u_t = 0 \rightarrow \boxed{u = f(x + at)}$

② $B_2 = aD^2 + fD + g \rightarrow$ similar as ①

③ $B_3 = aD^3 + fD + g \rightarrow [L, B_3]y = D^2[(aD^3 + fD + g)y] - u[(-\dots)y] - B_3 Ly$

let a_i const

$$= \cancel{aD^3y} + \underbrace{D^2(gy)}_{D(g_x y + g y_x)} + \underbrace{D^2(fy_x)}_{-a u D^3y - f u D^2y - g u D^2y} - \underbrace{u B_3 y}_{-a D^3y + \boxed{a D^3(uy)}} - \underbrace{B_3(y_{xx} - u y)}_{-f D^3y + f D(uy) - g D^2y + g y_x}$$

$$\underbrace{g_{xx}y + \underline{2g_x y_x + g y_{xx}}}_{= f_{xx}y_x + \underline{2f_x y_{xx}} + f y_{xxx}} \quad \underbrace{f u_x y + f u y_x}_{\text{cancel}}$$

$$a D^3(uy) = a D^2(u_x y + u y_x) = a D(u_{xx}y + 2u_x y_x + u y_{xx})$$

$$= a(u_{xxx}y + \underline{3u_{xx}y_x} + 3u_x y_{xx} + \cancel{u y_{xxx}})$$

$$\therefore [L, B_3]y = y_{xx} \left(3a u_x + 2f_x \right) + y_x \left(\overset{a}{3}u_{xx} + f_{xx} + 2g_x \right) + y \left(a u_{xxx} + f u_x + g_{xx} \right)$$

Choose f, g $\ni$ $3a u_{xx} + f_{xx} + 2g_x = 0 \rightarrow \underbrace{3a u_x + f_x + 2g}_{= C_2}$

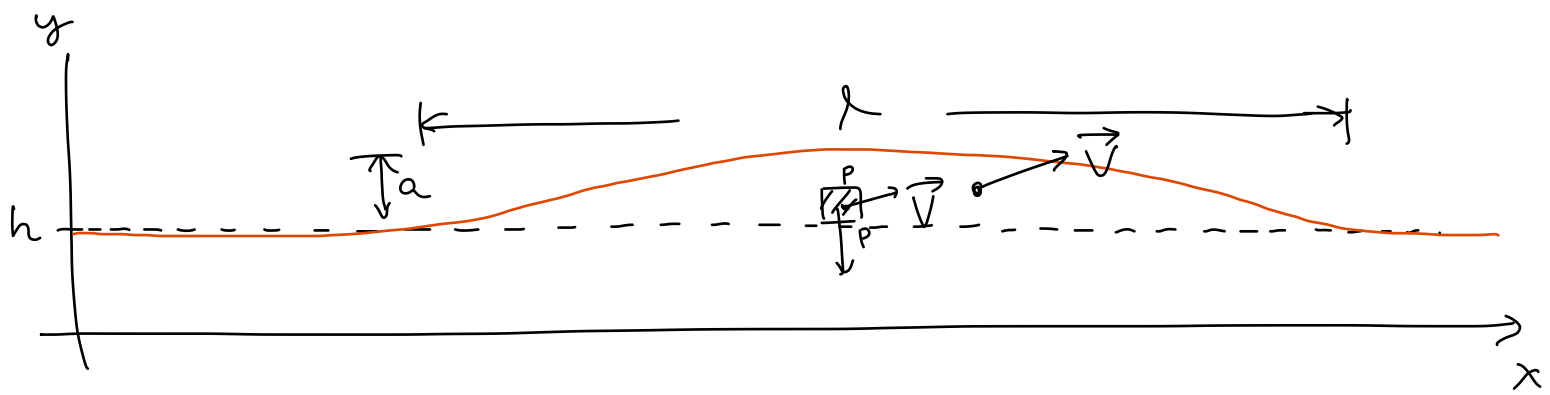
$$3a u_x + 2f_x = 0 \Rightarrow -2f_x$$

$$\left(f = -\frac{3}{2}a u + C_1, g = -\frac{3a}{4}u_x + C_2 \right)$$

$$\Rightarrow a u_{xxx} + \underbrace{f u_x}_{-\frac{3}{2}a u u_x + C_1 u_x} + \underbrace{g_{xx}}_{-\frac{3}{4}a u_{xxx}} = u_t \quad \text{for } \lambda_t = 0 \quad \text{set } C_1 = 0$$

$$\rightarrow \frac{a}{4} (u_{xxx} - 6u u_x) = u_t \quad \text{let } a = -4$$

$$\Rightarrow \boxed{u_t - 6u u_x + u_{xxx} = 0} \quad \text{KdV}$$



$$\lambda \gg h \gg a$$

$$\vec{V} = \vec{i} u + \vec{j} v \quad \text{no rotation} \quad \vec{\nabla} \times \vec{V} = 0 \rightarrow \vec{V} = \vec{\nabla} \phi$$

$$\text{Continuity} \quad \rho_t + \vec{\nabla} \cdot (\rho \vec{V}) = 0 \rightarrow \vec{\nabla} \cdot \vec{V} = 0 \quad \therefore \nabla^2 \phi = 0$$

$$y = y_1 = h + \eta(x, t) \rightarrow \frac{dy_1}{dt} = \eta_t + \eta_x \underbrace{\frac{dx_1}{dt}}_{u_1} = v_1 = \phi_y, \quad u_1 = \phi_x$$

$$v_1 = \eta_t + \eta_x u_1 \rightarrow \underline{\underline{\phi_y = \eta_t + \eta_x \phi_x}}$$

EoM

$$= \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \vec{\nabla}) \vec{V}$$

$$\underbrace{(dM)}_{\rho_0 V_0} \frac{d\vec{V}}{dt} = -(dM) g \vec{j} - V_0 \vec{\nabla} p \rightarrow \frac{d\vec{V}}{dt} = -\frac{1}{\rho_0} \vec{\nabla} p - g \vec{j}$$

$$\vec{V} = \vec{\nabla} \phi \leadsto \vec{\nabla} \phi_t = -\frac{1}{\rho_0} \vec{\nabla} p - g \vec{\nabla} y - \frac{1}{2} \vec{\nabla} v^2$$

$$(\vec{V} \cdot \vec{\nabla}) \vec{V} = \frac{1}{2} \vec{\nabla} v^2 - \underbrace{\vec{V} \times (\vec{\nabla} \times \vec{V})}_{=0}$$

$$\downarrow$$

$$\phi_t + \underbrace{\frac{1}{2} (\vec{\nabla} \phi)^2}_{\frac{1}{2} (u^2 + v^2)} + \frac{p}{\rho_0} + g y = 0$$

$\omega y + t$

$h + \eta$

$$\text{if assume } p = 0 \quad \& \quad \frac{\partial}{\partial x}$$

$$u_t + u u_x + v v_x + g \eta_x = 0$$

$$\text{Approx: } \phi(x, y, t) = \sum_{n=0}^{\infty} y^n \phi_n(x, t) \rightarrow \nabla^2 \phi = 0 \rightarrow \phi_{nxx} + (n+1)(n+2) \phi_{n+2} = 0$$

$$\text{B.C. } \phi_y(x, 0) = 0 \rightarrow \phi_1 = 0 \rightarrow \phi_{2n+1} = 0$$

$$\phi = \phi_0 - \frac{1}{2} y^2 \phi_{0xx} + \frac{1}{24} y^4 \phi_{0xxxx} + \dots$$

$$u = \phi_x = f - \frac{1}{2} y^2 f_{xx} + \dots \quad f \equiv \phi_{0x}$$

$$v = \phi_y = -y f_x + \frac{1}{6} y^3 f_{xxx} + \dots$$

leading term

$$v \approx \eta_t$$

$$\eta, u, u_x, v, v_x \ll 1$$

$$u_t + g \eta_x \approx 0$$

$$f \approx u \rightarrow f_t \approx -g \eta_x \quad -h f_x \approx \eta_t \Rightarrow \eta_{tt} - \overset{gh}{\uparrow} c_0^2 \eta_{xx} = 0$$

(wave equation)

$$u = -g \int \eta_x dx = \frac{g}{c_0} \eta = \epsilon c_0 e^{i(kx - \omega t)} \quad \eta = a e^{i(kx - \omega t)}$$

$$\left(\begin{array}{l} \frac{g}{c_0} a = \epsilon c_0 \\ a = \frac{gh\epsilon}{g} = h\epsilon \end{array} \right) \quad (\omega = c_0 k) \quad \epsilon \equiv \frac{a}{h}$$

$$v = \eta_t = -i\omega \eta = -\frac{i c_0}{\lambda} \eta \quad \lambda \sim l \quad v_i \sim c_0 \frac{a}{l} = c_0 \frac{a}{h} \frac{h}{l} = c_0 \epsilon \delta$$

keep 1st order term

$$\eta_t \approx -h f_x + \frac{1}{6} h^3 f_{xxx} \rightarrow \eta_{tx} = -h f_{xx} + \frac{1}{6} h^3 f_{xxxx}$$

$$u_t \approx f_t - \frac{1}{2} h^2 f_{xxt} \approx -g \eta_x \rightarrow \eta_{xt} = -\frac{1}{g} (f_{tt} - \frac{1}{2} h^2 f_{xxtt})$$

$$\therefore f_{tt} - c_0^2 f_{xx} + \frac{1}{6} c_0^2 h^2 f_{xxxx} - \frac{1}{2} h^2 f_{xxtt} = 0$$

$$f \sim e^{i(kx - \omega t)}$$

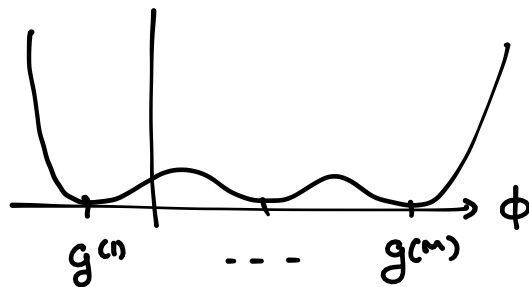
$$-\omega^2 + c_0^2 k^2 + \frac{1}{6} c_0^2 h^2 k^4 - \frac{1}{2} h^2 k^2 \omega^2 = 0$$

$$\omega^2 = \frac{c_0^2 k^2 \left(1 + \frac{1}{6} h^2 k^2 \right)}{1 + \frac{1}{2} h^2 k^2}$$

Solitary wave

$$\mathcal{L} = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \phi'^2 - U(\phi) \rightarrow \square \phi = -\frac{\partial U}{\partial \phi}$$

$$E = \int_{-\infty}^{\infty} dx \left[\frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \phi'^2 + U(\phi) \right]$$



$$E=0 \quad \text{if} \quad \phi(x,t) = g^{(i)} = \text{const.} \leftarrow \text{vacuum. (ground)}$$

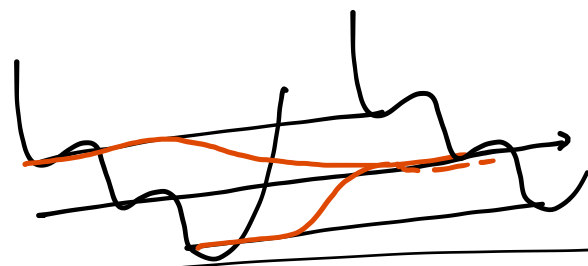
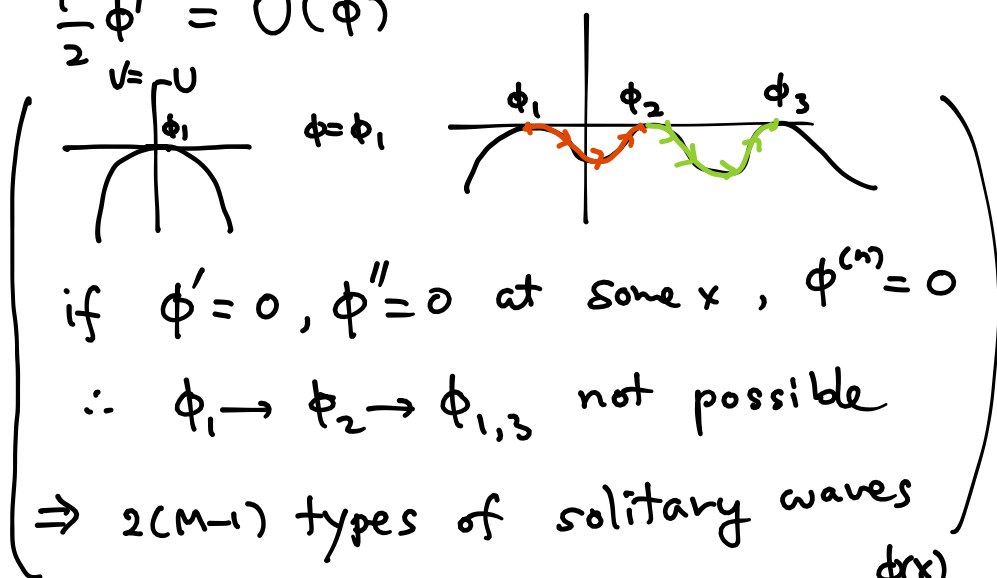
Static solution

$$\text{let } \dot{\phi} = 0 \rightarrow -\phi'' = -U'(\phi) \rightarrow \phi'' = \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial U}{\partial \phi}$$

$E > 0$ solution

for finite E ; as $x \rightarrow \pm \infty$, $\phi \rightarrow g^{(i)}$
 $\phi' \rightarrow 0$

$$\frac{1}{2} \phi'^2 = U(\phi)$$

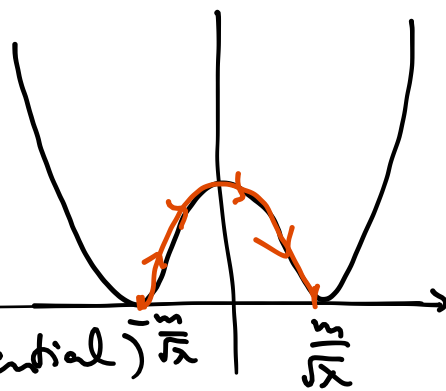


Analogy with Mechanics

"like Newton's law
 $x \rightarrow \text{time}$, $-U \rightarrow \text{potential}$
 $\mathcal{W} = \frac{1}{2} \phi'^2 - U = 0$

"Both initial & final conditions"

$$\frac{d\phi}{dx} = \pm \sqrt{2U(\phi)} \rightarrow x - x_0 = \pm \int_{\phi(x_0)}^{\phi(x)} \frac{d\phi}{\sqrt{2U(\phi)}}$$



" ϕ^4 " - "kink" : $U(\phi) = \frac{1}{4} \lambda (\phi^2 - \frac{m^2}{\lambda})^2$ (Higgs potential)

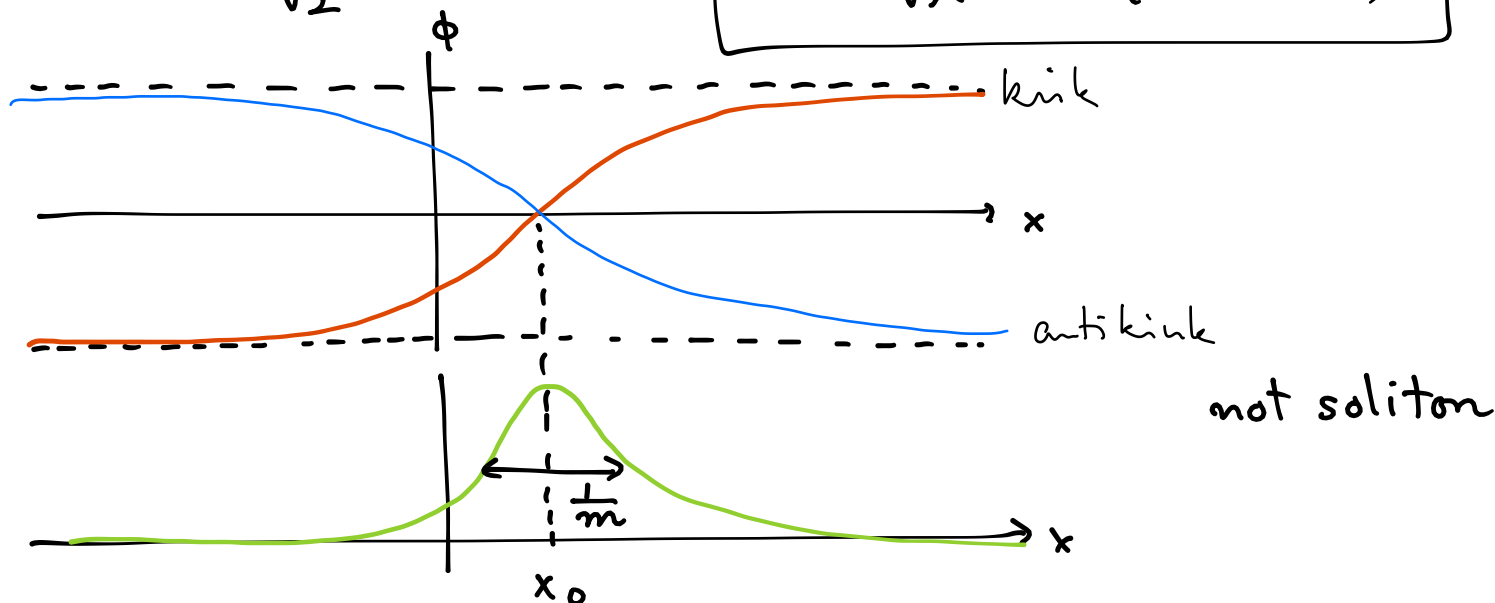
$$\ddot{\phi} - \phi'' = m^2 \phi - \lambda \phi^3$$

$$\text{static solution: } \dot{\phi} = 0 \rightarrow x - x_0 = \pm \int_{\phi(x_0)}^{\phi(x)} \frac{d\phi}{\sqrt{\frac{\lambda}{2} (\phi^2 - \frac{m^2}{\lambda})}}$$

$$\phi \equiv \sqrt{\frac{m^2}{\lambda}} \tanh \theta \rightarrow d\phi = \frac{m}{\sqrt{\lambda}} \frac{1}{\cosh^3 \theta} d\theta$$

$$x - x_0 = \pm \int \frac{\frac{m}{\sqrt{\lambda}} \frac{1}{\cosh^3 \theta} d\theta}{\sqrt{\frac{\lambda}{2}} \frac{m^2}{\lambda} \frac{1}{\cosh^2 \theta}} = \pm \frac{\sqrt{2}}{m} (\theta - \theta_0) \quad \text{let } \theta_0 \equiv 0$$

$$\therefore \theta = \pm \frac{m}{\sqrt{2}} (x - x_0) \rightarrow \boxed{\phi^{(\pm)} = \pm \frac{m}{\sqrt{\lambda}} \tanh\left(\frac{m}{\sqrt{2}} (x - x_0)\right)}$$

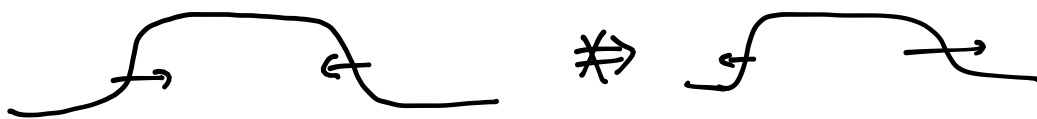


$$\mathcal{E}(x) = \frac{1}{2} \phi'^2 + U(\phi) = 2 U(\phi) =$$

$$E = \int_{-\infty}^{\infty} \mathcal{E}(x) dx = \frac{m^4}{2\lambda} \frac{\sqrt{2}}{m} \int_{-\infty}^{\infty} \frac{da}{\cosh^4 a} = \frac{m^3}{\sqrt{2}\lambda} \cdot \frac{4}{3} = \frac{2\sqrt{2}}{3} \frac{m^3}{\lambda}$$

- no two-kink states are possible.

- kink-antikink
not soliton



Moving kink: by Lorentz transformation

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma + \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$

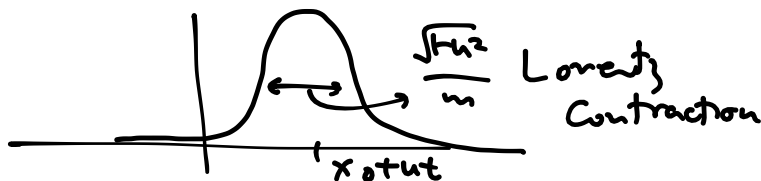
$$\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \gamma - \gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix}$$

$$\gamma^2 - \gamma^2 \beta^2 = 1$$

$$\Rightarrow x = \gamma (x' - \beta t')$$

$$\phi_u(x, t) = \frac{m}{\sqrt{\lambda}} \tanh \left[\frac{m}{\sqrt{2}} \frac{x - x_0 - ut}{\sqrt{1 - u^2}} \right] \quad (-1 < u < 1)$$

at some t



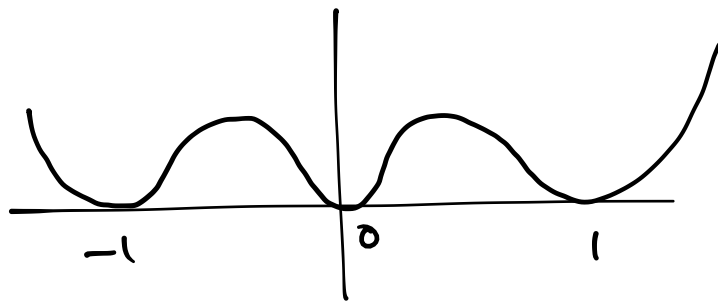
$$E[\phi_u] = \int dx \left[\frac{1}{2} \dot{\phi}_u^2 + \frac{1}{2} \phi_u'^2 + U(\phi_u) \right] = \frac{m^4}{4\lambda} \left(1 + \frac{u^2}{1-u^2} + \frac{1}{1-u^2} \right) \int \underbrace{\text{sech}^4(\dots) dx}_{\frac{\sqrt{2}\sqrt{1-u^2}}{m} \cdot \frac{4}{3}}$$

$$\left(\phi_u(x,t) = \frac{m}{\sqrt{\lambda}} \tanh \left[\frac{m}{\sqrt{2}} \frac{x-x_0-ut}{\sqrt{1-u^2}} \right] \right. \quad \begin{aligned} U(\phi_u) &= \frac{1}{4} \lambda \left(\frac{m}{x} \right)^2 \text{sech}^4(\dots) \\ \dot{\phi}_u^2 &= \left(\frac{m^2}{\sqrt{2}\lambda} \frac{u}{\sqrt{1-u^2}} \right)^2 \text{sech}^4(\dots) \\ \phi_u'^2 &= \left(\frac{m^2}{\sqrt{2}\lambda} \frac{1}{\sqrt{1-u^2}} \right)^2 \text{sech}^4(\dots) \end{aligned}$$

$$= \frac{m^4}{2\lambda} \frac{1}{1-u^2} \frac{4\sqrt{2}}{3^{\frac{2}{3}}} \sqrt{1-u^2} = \frac{m^3 2\sqrt{2}}{3\lambda} \frac{1}{\sqrt{1-u^2}} = \frac{M_{cl}}{\sqrt{1-u^2}} > M_{cl}$$

H.W. $U(\phi) = \frac{1}{2} \phi^2 (\phi^2 - 1)^2$

Find $\phi(x)$ static solution.



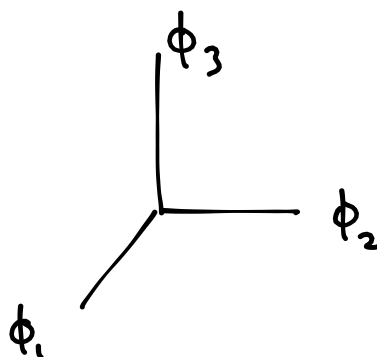
Many fields

$$\mathcal{L} = \sum_{i=1}^N \frac{1}{2} (\partial_\mu \phi_i)^2 - U(\{\phi_i\})$$

Analogy of Mechanics

$-U(\phi_i)$: potential

$$\Rightarrow \square \phi_i = -\frac{\partial U}{\partial \phi_i} \Rightarrow \phi_i'' = \frac{\partial U}{\partial \phi_i}$$

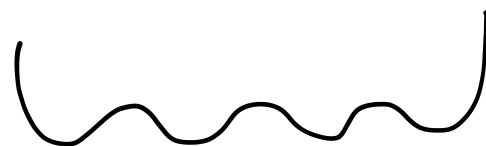


Omit

Topological Indices

$\lim_{x \rightarrow \infty} \phi(x,t_0) = \phi_1$ for any t_0

$\lim_{x \rightarrow -\infty} \phi(x,t_0) = \phi_2$ "



$$Q = \frac{\sqrt{\lambda}}{m} [\phi(\infty) - \phi(-\infty)] \text{ topo. charge}$$

$$k^\mu = \frac{\sqrt{\lambda}}{m} \epsilon^{\mu\nu} \partial_\nu \phi \rightarrow \partial_\mu k^\mu = \epsilon^{\mu\nu} \partial_\mu \partial_\nu \phi = 0$$

$$Q = \int dx k_0 = \frac{\sqrt{\lambda}}{m} \int dx \phi' = \frac{\sqrt{\lambda}}{m} (\phi(\infty) - \phi(-\infty)) \neq 0 \text{ "topological"}$$

top. change vs. SSB

SG soliton

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 + \frac{m^4}{\lambda} \left[\cos\left(\frac{\sqrt{\lambda}}{m} \phi\right) - 1 \right]$$

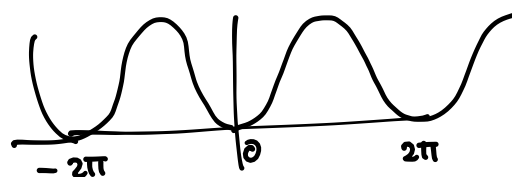
$$\cong \frac{1}{2} (\partial_\mu \phi)^2 - \frac{m^2}{2} \phi^2 + \frac{\lambda}{4!} \phi^4 - \frac{\lambda^2}{m^2 6!} \phi^6 + \dots$$

$$\text{EOM: } \square \phi + \frac{m^3}{\sqrt{\lambda}} \sin\left[\frac{\sqrt{\lambda}}{m} \phi\right] = 0$$

$$m_X = \bar{X}, m_t = \bar{t}, \bar{\phi} = \frac{\sqrt{\lambda}}{m} \phi \rightarrow \bar{\mathcal{L}} = \frac{m^4}{\lambda} \left[\frac{1}{2} (\partial_\mu \bar{\phi})^2 + \cos \bar{\phi} - 1 \right]$$

$$E = \frac{m^3}{\lambda} \int d\bar{x} \left[\frac{1}{2} \left(\frac{\partial \bar{\phi}}{\partial \bar{t}} \right)^2 + \frac{1}{2} \left(\frac{\partial \bar{\phi}}{\partial \bar{x}} \right)^2 + \underbrace{1 - \cos \bar{\phi}}_{U(\bar{\phi})} \right] \quad \square \bar{\phi} + \sin \bar{\phi} = 0$$

$$\bar{\phi} = 2\pi N, \quad N = -\infty, \dots, \infty$$



$$Q = N_1 - N_2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\bar{x} \partial_{\bar{x}} \bar{\phi}$$

Static solution

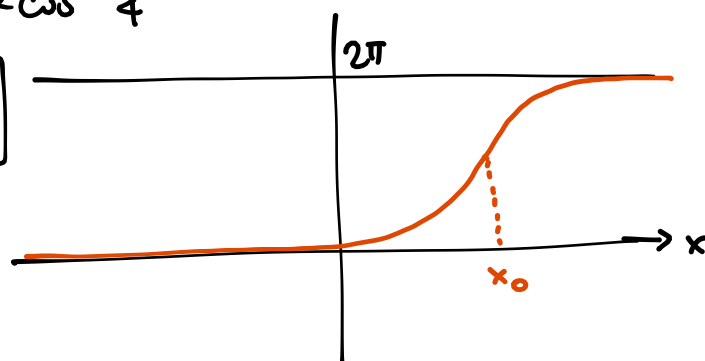
$$\begin{aligned} \bar{x} - \bar{x}_0 &= \pm \int_{\bar{\phi}_0}^{\bar{\phi}} \frac{d\phi}{\sqrt{2U}} = \pm \int_{\bar{\phi}_0}^{\bar{\phi}} \frac{d\phi}{2 \sin \frac{\phi}{2}} \\ &= \pm \int \frac{-2d\alpha}{2(1-\alpha^2)} = \pm \frac{1}{2} \int_{\alpha}^{\alpha_0} \left(\frac{1}{1-\alpha} + \frac{1}{1+\alpha} \right) d\alpha = \pm \frac{1}{2} \log \frac{1-\alpha}{1+\alpha} \end{aligned}$$

$$\sin^2 \frac{\phi}{2} = 1 - \cos^2 \frac{\phi}{2}$$

$$\cos \frac{\phi}{2} \equiv \alpha \quad d\alpha = -\frac{1}{2} \sin \frac{\phi}{2} d\phi$$

$$\therefore e^{\pm 2(\bar{x} - \bar{x}_0)} = \frac{1 - \cos \frac{\phi}{2}}{1 + \cos \frac{\phi}{2}} = \frac{2 \sin^2 \frac{\bar{\phi}}{4}}{2 \cos^2 \frac{\bar{\phi}}{4}} = \tan^2 \frac{\bar{\phi}}{4}$$

$$\Rightarrow \bar{\phi} = 4 \tan \left[e^{(x - \bar{x}_0)} \right]$$



H.W.

$$\mathcal{L} = \frac{1}{2} m l^2 \dot{\theta}^2 + m g l \cos \theta$$

$$m l^2 \ddot{\theta} = -m g l \sin \theta \rightarrow \ddot{\theta} + \frac{g}{l} \sin \theta = 0 \quad \frac{g}{l} \equiv \omega_0^2$$

$$E = \frac{1}{2} m l^2 \dot{\theta}^2 - m g l \cos \theta \rightarrow \dot{\theta} = \pm \sqrt{2 \omega_0^2 (\cos \theta - \cos \theta_0)} = \frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = \omega_0 \sin \frac{\theta_0}{2} \sqrt{1 - \frac{1}{\sin^2 \frac{\theta_0}{2}} \sin^2 \frac{\theta}{2}}$$

$$\frac{1}{\sqrt{1 - \frac{1}{\sin^2 \frac{\theta_0}{2}} \sin^2 \frac{\theta}{2}}} d\left(\frac{\theta}{2}\right) = \frac{\omega_0}{2} \sin \frac{\theta_0}{2} \int_0^t dt$$

$$\equiv K\left(\sin^2 \frac{\theta_0}{2}; \frac{\theta}{2}\right) = \frac{\omega_0}{2} \sin \frac{\theta_0}{2} t //$$

Felsager

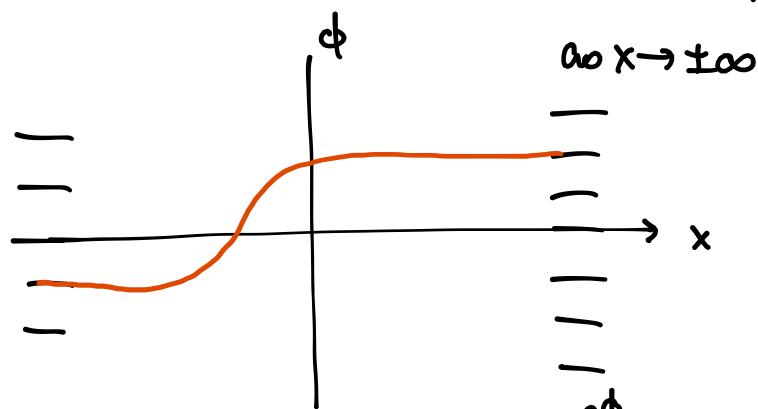
Ground States for nonpert. sector

$$H_{\text{static}}[\phi] = \int_{-\infty}^{\infty} \left[\frac{1}{2} (\phi')^2 + U(\phi) \right] dx \xrightarrow{\text{minimum}} \phi'' = U'[\phi]$$

$\downarrow$

$$(\phi')^2 = 2U[\phi]$$

$$\therefore \phi' = \pm \sqrt{2U(\phi)} \quad \therefore \phi \rightarrow \phi_n \leftarrow \text{minima of } U$$



$$\phi' = \frac{d\phi}{dx} = \sqrt{2U(\phi)} \rightarrow \int_{\phi_0}^{\phi} \frac{d\phi}{\sqrt{2U(\phi)}} = \int_{x_0}^x dx = x - x_0$$

Lorentz boost: $\phi(x, t) \rightarrow \phi(x', t') = \phi(\gamma(x - vt), \gamma(t - vx))$

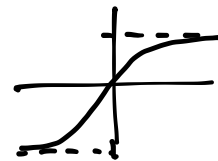
static $\phi(x, t) = \phi[\gamma(x - vt)]$

$$= \frac{1}{2} \int \left[\phi' \mp \sqrt{2U} \right]^2 dx \quad \pm \underbrace{\int_{-\infty}^{\infty} \phi' \sqrt{2U} dx}_{\int \sqrt{2U} d\phi = \text{constant topological term}}$$

$H_{\text{static}} \supseteq$ top. terms

when $\phi' = \pm \sqrt{2U}$

$$\phi'' = \phi(\phi^2 - 1) \rightarrow x \rightarrow \pm\infty$$



$$\phi' = \frac{d\phi}{dx} = \frac{d\phi}{dy} \frac{dy}{dx}$$

$$y = \tanh x$$

$$\frac{dy}{dx} = \frac{1}{\cosh^2 x}$$

$$\phi'' = \frac{d}{dx} \left(\frac{d\phi}{dy} \frac{1}{\cosh^2 x} \right) = \frac{d^2\phi}{dy^2} \underbrace{\frac{dy}{dx} \frac{1}{\cosh^2 x}}_{\frac{1}{\cosh^4 x}} + \frac{d\phi}{dy} \underbrace{\left(\frac{d}{dx} \left(\frac{1}{\cosh^2 x} \right) \right)}_{-2 \frac{1}{\cosh^3 x} \sinh x}$$

$$y^2 = \tanh^2 x$$

$$1 - y^2 = 1 - \tanh^2 x = \frac{1}{\cosh^2 x}$$

$$\frac{1}{\cosh^4 x} = (1 - y^2)^2$$

$$-2 \frac{1}{\cosh^3 x} \sinh x = -2(1 - y^2)y$$

$$\Rightarrow \left[\frac{d^2\phi}{dy^2} (1 - y^2)^2 - 2 \frac{d\phi}{dy} y (1 - y^2) = \phi(\phi^2 - 1) \right]$$

Bäcklund transf

multi-soliton

introduce light-cone coord. ($c \equiv 1$)

$$x^{\pm} = \frac{1}{2}(x \pm t) \quad \partial_{\pm} = \frac{\partial x}{\partial x_{\pm}} \partial_{x_{\pm}} + \frac{\partial t}{\partial x_{\pm}} \partial_t = \partial_{x \pm} \partial_t$$

$$\text{SG E of M: } \partial_+ \partial_- \phi = (\partial_x^2 - \partial_t^2) \phi = \frac{\mu^2}{\lambda} \sin \lambda \phi$$

$$\text{let } \mu = \lambda \equiv 1 \quad \boxed{\partial_+ \partial_- \phi = \sin \phi}$$

2nd order DE $\rightarrow$ 1st coupled DE.

$$\text{let } \underbrace{\partial_+ \left(\frac{\phi_1 - \phi_0}{2} \right)}_{\phi^-} = a \sin \left(\underbrace{\frac{\phi_1 + \phi_0}{2}}_{\phi^+} \right), \quad \partial_- \left(\frac{\phi_1 + \phi_0}{2} \right) = \frac{1}{a} \sin \left(\frac{\phi_1 - \phi_0}{2} \right)$$

$$\partial_+ \phi^- = a \sin \phi^+, \quad \partial_- \phi^+ = \frac{1}{a} \sin \phi^-$$

$$\partial_+ \partial_- \phi^- = a \cos \phi^+ \partial_- \phi^+ = \cos \phi^+ \sin \phi^-$$

$$\partial_+ \partial_- \phi^+ = \frac{1}{a} \cos \phi^- \partial_+ \phi^- = \cos \phi^- \sin \phi^+$$

$$\partial_+ \partial_- \begin{pmatrix} \phi_1 \\ -\phi_0 \end{pmatrix} = \sin \begin{pmatrix} \phi_1 \\ -\phi_0 \end{pmatrix} \rightarrow \phi_0 \text{ \& \& } \phi_1 \text{ are sol.}$$

If ϕ_0 is a sol, ϕ_1 is also sol.

notice: a is a free parameter.

$$\textcircled{1} \text{ let } \phi_0 = 0, \phi_1 = \phi : \begin{cases} \partial_+ \phi = 2a \sin \frac{\phi}{2} \\ \partial_- \phi = \frac{2}{a} \sin \frac{\phi}{2} \end{cases}$$

$$\partial_+ \phi = 2a \sin \frac{\phi}{2}$$

$$\partial_- \phi = \frac{2}{a} \sin \frac{\phi}{2}$$

let

$$ax_+ + \frac{1}{a}x_- \equiv \xi$$

$$ax_+ - \frac{1}{a}x_- \equiv \eta$$

$$x_+ = \frac{1}{2a}(\xi + \eta)$$

$$x_- = \frac{a}{2}(\xi - \eta)$$

$$\partial_+ = \frac{\partial}{\partial x_+} = a(\partial_\xi + \partial_\eta)$$

$$\partial_- = \frac{\partial}{\partial x_-} = \frac{1}{a}(\partial_\xi - \partial_\eta)$$

$$\left. \begin{aligned} (\partial_\xi + \partial_\eta)\phi &= 2 \sin \frac{\phi}{2} \\ (\partial_\xi - \partial_\eta)\phi &= \frac{2}{a} \sin \frac{\phi}{2} \end{aligned} \right\}$$

$$\rightarrow \partial_\eta \phi = 0$$

$$\partial_\xi \phi = 2 \sin \frac{\phi}{2}$$

$$\phi = \phi(\xi)$$

$$\tan \frac{\phi}{4} = e^{\xi - \xi_0}$$

$$= e^{\frac{1}{2}(a + \frac{1}{a})x + \frac{1}{2}(a - \frac{1}{a})t - \xi_0}$$

$$= e^{\frac{1}{2}(a + \frac{1}{a})(x - vt) - \xi_0}$$

$$\uparrow \frac{\frac{1}{a} - a}{\frac{1}{a} + a}$$

$$1 - v^2 = 1 - \frac{(\frac{1}{a} - a)^2}{(\frac{1}{a} + a)^2} = \left(\frac{2}{\frac{1}{a} + a}\right)^2$$

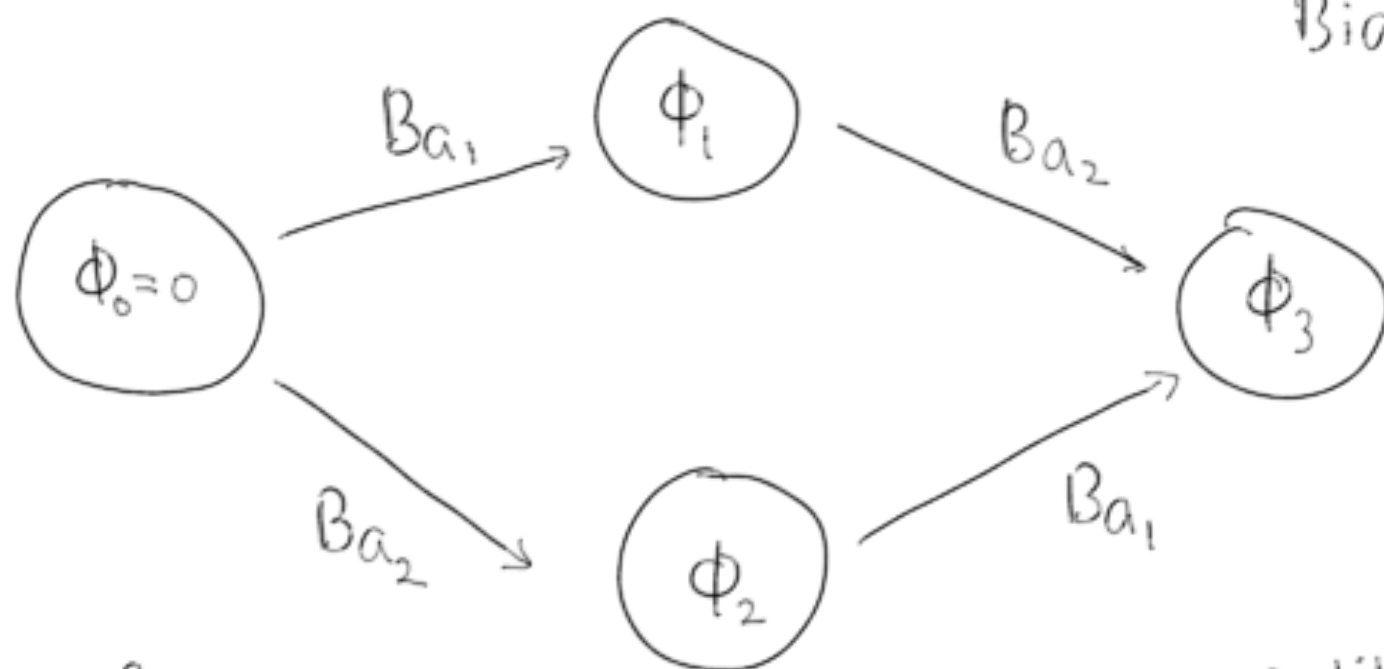
$$\gamma = \frac{1}{\sqrt{1 - v^2}} = \frac{1}{2} \left(a + \frac{1}{a}\right)$$

$$\phi_1 = 4 \tan^{-1} \left(e^{\gamma_a(x - vt) - \xi_0} \right)$$

$$\equiv B_a(\phi_0)$$

a : boost. (moving 1-soliton solution)

Bianchi Id.



Bianchi Id.

$$B_{a_2}[B_{a_1}[\phi_0]] = B_{a_1}[B_{a_2}[\phi_0]]$$

Show.

Multi-Soliton sol.

$$\cos \phi = 1 - 2 \partial_x \partial^N \ln(\det M),$$

$$M_{ij} = \frac{2}{a_i + a_j} \cosh \frac{\theta_i + \theta_j}{2},$$

$$a_i^2 = \frac{1-v_i}{1+v_i}, \quad \theta_i = \pm \gamma_i (x - v_i t - \xi_i)$$

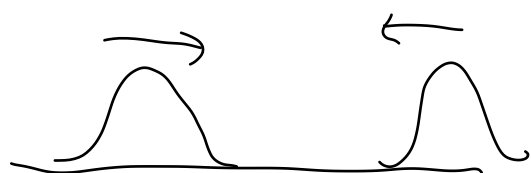
explicit 2-soliton sol.

$$\tan \frac{\phi}{4} = \frac{a_2 + a_1}{a_2 - a_1} \frac{\sinh \left(\frac{1}{2} (\xi_2 - \xi_1 - \xi_0^{(2)} + \xi_0^{(1)}) \right)}{\cosh \left(\frac{1}{2} (\xi_2 - \xi_1 - \xi_0^{(2)} - \xi_0^{(1)}) \right)}$$

Make movie with Mathematica

$$\tan \frac{\phi_{12} - \phi_0}{4} = \frac{a_1 + a_2}{a_1 - a_2} \frac{\sinh \frac{y_1 - y_2}{2}}{\cosh \frac{y_1 + y_2}{2}} = \frac{1}{v} \frac{\sinh \gamma v t}{\cosh \gamma v x} \quad \text{with } a_2 = -2 \gamma v t$$

let $a_1 = \frac{1}{a_2}$ $y_1 - y_2 = \frac{a_1 + \frac{1}{a_1}}{2} \left((x - vt) - (x + vt) \right)$



$y_1 + y_2 = \quad \quad \quad + \quad \quad \quad)$

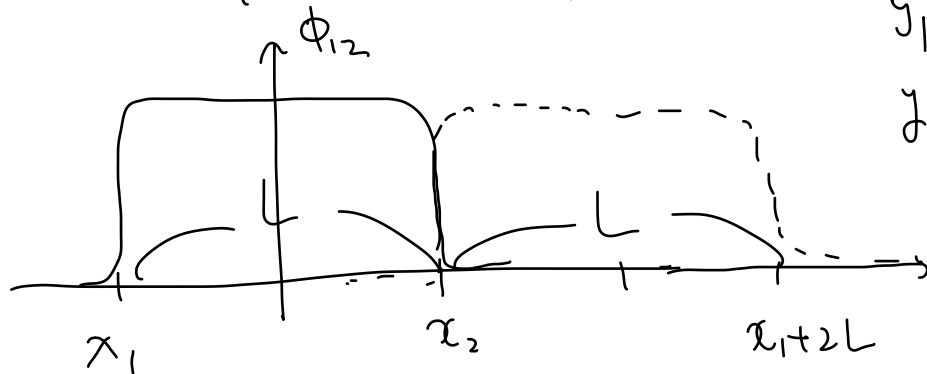
$t \rightarrow +\infty \quad \frac{1}{v} \sinh \gamma v t \approx \frac{e^{\gamma v t}}{2v} = \frac{1}{2} e^{\gamma v t - \frac{\ln v}{\gamma v}}$
 $t \rightarrow -\infty \quad \frac{1}{v} \sinh \gamma v t \approx -\frac{e^{-\gamma v t}}{2v} \quad \Delta t = \frac{2}{\gamma v} \ln v$

$$\boxed{\Delta t = \frac{2}{\gamma v} \ln v} \quad (< 0) = -\frac{1}{2} e^{-\gamma v (t + \frac{\ln v}{\gamma v})}$$

or $a_2 = 1$ $\tan \frac{\phi_{12}}{4} = \frac{a_1 + 1}{a_1 - 1} \frac{\sinh \frac{y_1 - y_2}{2}}{\cosh \frac{y_1 + y_2}{2}}$

$y_2 = x - x_2 \quad y_1 = \gamma (x - x_1 - vt)$

at $t=0$



$y_1 = \gamma (x - x_1)$

$y_2 = x - x_2$

at $x = x_1 \rightarrow y_1 \approx 0$

at $x = x_2 \rightarrow y_2 \approx 0$

at $x = x_2 \rightarrow y_1 = 0$

at $t=T$

$y_1 = \gamma (x - x_1 - vT)$

at $x = x_1 + 2L$

$-\frac{y_1 - L}{2}$

$y_2 = x - x_2 \rightarrow x = x_1 + L$

$y_2 = x_1 - x_2 + 2L = L$

$-\frac{a_1 + 1}{a_1 - 1} \frac{e^{-\frac{y_1 - L}{2}}}{e^{\frac{y_1 + L}{2}}} = -\frac{a_1 + 1}{a_1 - 1} e^{-\gamma (2L - vT)}$

$\sim \frac{a_1 + 1}{a_1 - 1} e^{-\gamma (2L - vT)}$

$\therefore y_1 = \gamma (2L - vT) \approx \mathcal{O}(1)$

$$(cf) \quad \tan \frac{\phi_1}{4} = e^{\gamma(x - vt - x_1)}$$

$$t=0 \quad e^{\gamma(x - x_1)} \quad \int_{x=x_1}^{\quad}$$

$$T_1 \tan \frac{\phi_1}{4} = e^{\gamma(2L - vT_1)}$$

$$(cf) \quad \tan \frac{\phi_{12}}{4} = -\frac{a_{i+1}}{a_i - 1} e^{\gamma(2L - vT)}$$

$$\gamma = \frac{a_i + \frac{1}{a_i}}{2}$$

$$a_i^2 - 2a_i\gamma + 1 = 0$$

$$a_i = \gamma - \sqrt{\gamma^2 - 1}$$

$$\frac{a_i + 1}{a_i - 1} = \frac{\gamma + 1 - \sqrt{\gamma^2 - 1}}{\gamma - 1 - \sqrt{\gamma^2 - 1}} = \frac{\sqrt{\frac{\gamma+1}{\gamma-1}}}{\sqrt{\frac{\gamma-1}{\gamma+1}}} \frac{\sqrt{\gamma+1} - \sqrt{\gamma-1}}{\sqrt{\gamma-1} - \sqrt{\gamma+1}} = -\sqrt{\frac{\gamma+1}{\gamma-1}}$$

$$e^{-\gamma \left(2L - vT - \frac{1}{\gamma} \log \sqrt{\frac{\gamma+1}{\gamma-1}} \right)}$$

$$v \left(T + \frac{1}{\gamma v} \log \sqrt{\frac{\gamma+1}{\gamma-1}} \right)$$

$$\therefore \boxed{T - T_1 = -\frac{1}{2\gamma v} \log \frac{\gamma+1}{\gamma-1}}$$

Stability shows

let ϕ_0 is a soliton solution.

what if $\phi = \phi_0 + \eta(x, t)$, $|\eta| \ll 1$

E of M: $\partial_\mu \partial^\mu \phi = U'(\phi)$

$$\partial_\mu \partial^\mu \phi_0 = U'(\phi_0) \rightarrow \boxed{\partial_\mu \partial^\mu \eta = U''(\phi_0) \eta}$$

let $\eta = \sum_k e^{-i\omega_k t} \phi_k(x)$

$$\sum_k \left(\omega_k^2 + \frac{\partial^2}{\partial x^2} \right) \phi_k(x) e^{-i\omega_k t} = \sum_k U''(\phi_0(x)) \phi_k e^{-i\omega_k t}$$

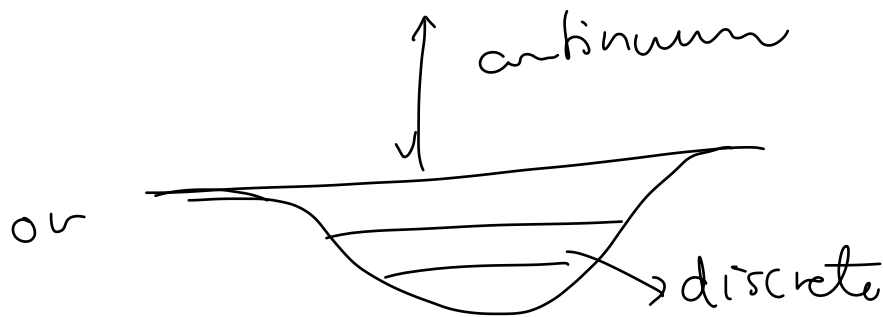
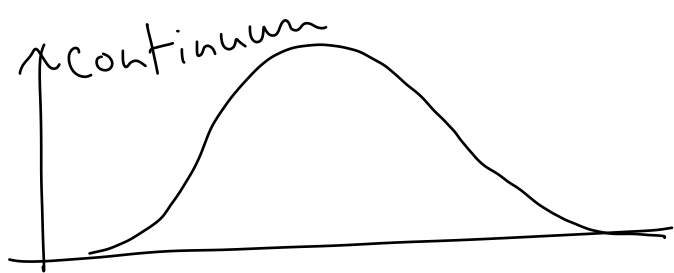
$$\Rightarrow \left[-\frac{\partial^2}{\partial x^2} + U''(\phi_0(x)) \right] \phi_k = \omega_k^2 \phi_k$$

a kind of Schrödinger eq. ↑ eigenvalues

if $\omega_k^2 < 0 \rightarrow e^{\pm \omega_k t} \rightarrow$ diverge
either in past or future

$\therefore$ Unstable

as $x \rightarrow \pm \infty$ $\phi_0 \rightarrow \text{const} \rightarrow U''(\phi_0) \rightarrow 0$



$$\frac{d^2 \phi_0}{dx^2} = U'(\phi_0(x))$$

$$\rightarrow \frac{d^3 \phi_0}{dx^3} = U''(\phi_0) \frac{d\phi_0}{dx}$$

$\rightarrow \frac{d\phi_0}{dx}$ is a solution of Schrödinger Eq. with $\lambda=0$

Derrick's theorem. (Static solution)

$$E = \underbrace{\frac{1}{2} \int \left(\frac{\partial \phi^a}{\partial x^i} \right) \left(\frac{\partial \phi^a}{\partial x^i} \right) d^D x}_{H_1} + \underbrace{\int U(\phi^a) d^D x}_{H_2}$$

$$\text{let } \phi_\lambda^a(x) = \phi^a(\lambda x)$$

$$E(\lambda) = \int d^D x \frac{1}{2} \left(\frac{\partial \phi_\lambda^a}{\partial x^i} \right)^2 d^D x + \int U(\phi_\lambda^a) d^D x$$

$$= \lambda^{2-D} H_1 + \lambda^{-D} H_2 \quad \lambda=1 \text{ is min.}$$

$$\therefore \frac{\partial E(\lambda)}{\partial \lambda} \Big|_{\lambda=1} = 0 \rightarrow \boxed{(2-D)H_1 - D H_2 = 0}$$

$D > 2$ not possible. ($\because H_1, H_2 \geq 0$)

$$D=1 \quad H_1 = H_2 \rightarrow \phi' = \pm \sqrt{2U} \text{ (our sol.)}$$

$$D=2 \quad H_2 = 0 \rightarrow \text{no potential.}$$

Zakharov-Shabat formulation

[first-order formulation]

nonlinear DE $\rightarrow$ coupled linear DE with consistent conditions

Lax pair

$$\left. \begin{aligned} L(t) \psi(t) &= -\lambda \psi(t) \\ \frac{\partial L}{\partial t} &= [B(t), L(t)] \end{aligned} \right\} \quad \& \quad \frac{\partial \psi}{\partial t} = B(t) \psi(t)$$

$$\rightarrow [B, L] \psi + L B \psi = B L \psi = -\lambda B \psi - \underbrace{\frac{\partial \lambda}{\partial t} \psi}_{\equiv 0}$$

$$\therefore \frac{\partial \lambda}{\partial t} = 0$$

$$\psi(t) = U(t) \psi(0) \rightarrow \frac{\partial \psi}{\partial t} = \frac{\partial U}{\partial t} U^{-1} \psi(t) = B(t) \psi(t)$$

$$U: \text{unitary} \quad U^\dagger U = 1 \rightarrow \partial_t U^\dagger U + U^\dagger \partial_t U = 0$$

if $\partial_t U \equiv B U$ $\xrightarrow{\text{}} \text{satisfied if } B^\dagger = -B$

$$U^\dagger(t) L(t) U(t) = L(0)$$

$$\rightarrow \underbrace{\partial_t U^\dagger} L U + U^\dagger \partial_t L U + U^\dagger L \underbrace{\partial_t U}_{B U} = 0$$

$$- U^\dagger B L U$$

$$\rightarrow U^\dagger (\partial_t L - [B, L]) U = 0$$

$$\text{or } \partial_t L = [B, L]$$

Lax pair

assume

$$L(t) \psi(t) = -\lambda \psi(t)$$

$$\& \quad \frac{\partial \psi}{\partial t} = B \psi \quad \text{w/ } \frac{\partial \lambda}{\partial t} = 0$$

$$\begin{array}{c} \downarrow \text{linear in } u \\ \frac{\partial L}{\partial t} = [B, L] \end{array} \quad \text{nonli in } u$$

Lax Eq

$$\text{let } \psi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$L(\psi) = \sigma_3 \partial_x \psi - g(t) \sigma_+ \psi + r(t) \sigma_- \psi$$

$$L(t) \psi = -\lambda \psi \quad \lambda = i\zeta$$

$$\sigma_3 \partial_x \psi = g \sigma_+ \psi - r \sigma_- \psi - i\zeta \psi$$

$$\partial_x \psi = g \sigma_3 \sigma_+ \psi - r \sigma_3 \sigma_- \psi - i\zeta \sigma_3 \psi$$

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \sigma_+ \quad \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -\sigma_-$$

$$= \left[\underbrace{g \sigma_+ + r \sigma_-}_{\text{indep of } \zeta} - i\zeta \sigma_3 \right] \psi$$

$$\text{let } \partial_t \psi = B(t) \psi(t) \equiv \underbrace{(P \sigma_+ + Q \sigma_- + R \sigma_3)}_{B \text{ (depends on } \zeta)} \psi$$

$$\partial_x \partial_t \psi = \partial_t \partial_x \psi$$

$$\Rightarrow R_x = gQ - rP$$

$$r_t = Q_x - 2rR - 2i\zeta Q$$

$$g_t = P_x + 2gR + 2i\zeta P$$

$$\left. \begin{array}{l} \text{DE. Eq. of soliton} \\ \uparrow \\ \text{Lax Eq.} \end{array} \right\}$$

$$L = \partial_x - A(x, \zeta) = \partial_x - g \sigma_+ - r \sigma_- + i\zeta \sigma_3$$

$$B = P \sigma_+ + Q \sigma_- + R \sigma_3$$

$$\partial_t L = [B, L]$$

$$\sigma_{\pm}, \sigma_3 = -\sigma_3 \sigma_{\pm}$$

$$[\sigma_+, \sigma_-] = \sigma_3$$

$$[B, L] = -Pr \sigma_3 - 2i\zeta P \sigma_+ + gQ \sigma_3 + \underbrace{2i\zeta Q \sigma_-}_{-P_x \sigma_+ - Q_x \sigma_- - R_x \sigma_3} - 2gR \sigma_+ + \underbrace{2rR \sigma_-}$$

$$r_t = Q_x - 2rR - 2i\zeta Q$$

$$g_t = P_x + 2gR + 2i\zeta P$$

$$\checkmark 0 = -R_x + gQ - rP$$

SG

$$r(x,t) = -g(x,t) = \frac{1}{2} \omega_x(x,t)$$

$$p = Q = \frac{i}{4\zeta} \sin \omega$$

$$gQ - rP = \frac{i \omega_x}{4\zeta} \sin \omega = R_x \rightarrow R = \frac{i}{4\zeta} \cos \omega$$

$$r_t = Q_x - 2rR - 2i\zeta Q \rightarrow \frac{1}{2} \omega_{xt} = \frac{i}{4\zeta} \omega_x \cos \omega - \frac{i \omega_x \omega}{4\zeta}$$

$$g_t = p_x + 2gR + 2i\zeta p + \frac{1}{2} \sin \omega$$

$$\left(-\frac{1}{2} \omega_{xt} = -\frac{1}{2} \sin \omega \right)$$

$$\underbrace{\omega_{xt} = \sin \omega}_{x: + \quad t: -} \quad \text{SGE.}$$

NSE

$$r = g^* = \sqrt{k} \psi$$

$$Q = \varepsilon_k p^* \quad (\varepsilon_k = \text{sgn}(k))$$

$$R = 2i\zeta^2 + i k |\psi|^2$$

$$\sqrt{k} \psi_t = i\sqrt{k} \psi_{xx} + \alpha \cancel{\psi_x}$$

$$- 2\sqrt{k} \psi (2i\zeta^2 + i k |\psi|^2) - 2i\zeta (i\sqrt{k} \psi_x + \alpha \psi)$$

$$\therefore \text{let } \alpha = -2\zeta \sqrt{k}$$

$$\therefore \underline{i \psi_t = -\psi_{xx} + 2k |\psi|^2 \psi}$$

$$gQ - rP = \pm (\sqrt{k} \psi^* Q - \sqrt{k} \psi Q^*)$$

$$= i k 2 \times |\psi|^2$$

$$= i k (\psi^* \psi_x + \psi \psi_x^*)$$

$$Q = i\sqrt{k} \psi_x + \alpha \psi$$

$$= i\sqrt{k} (\psi_{xx} - 2k |\psi|^2 \psi)$$

$$L = \frac{i}{2} \int dx (\psi^* \psi_t - \psi_t^* \psi) - H$$

$$H = \int dx (\psi_x^* \psi_x + \kappa |\psi|^4)$$

Poisson bracket

$$\{\psi(x,t), \psi^*(y,t)\} = -i\delta(x-y)$$

$$\left\{ \underbrace{\psi(x,t)}_{\partial_t \psi}, H \right\} = \int dy \left\{ \psi(x), \underbrace{\psi_x^* \psi_x(y)}_{\substack{\psi^* \psi \\ \uparrow \quad \uparrow}} + \kappa |\psi|^4(y) \right\}$$

$$\partial_y \left\{ \underbrace{\psi(x), \psi^*(y)}_{-i\delta} \right\} \psi_x(y)$$

$$= i \psi_{xx} + 2\kappa |\psi|^2 \psi (-i)$$

$$\underline{i \psi_t = -\psi_{xx} + 2\kappa |\psi|^2 \psi} \quad \checkmark$$

$\therefore$ NSE is a Hamiltonian system.

Inverse scattering (let $\kappa < 0$)

$$[\partial_x - i\sqrt{|\kappa|} (\psi^* \sigma_+ + \psi \sigma_-) + i\zeta \sigma_3] \psi = 0$$

$$\frac{\partial \psi}{\partial t} = B \psi = (P \sigma_+ + Q \sigma_- + R \sigma_3) \psi$$

$$P = \sqrt{|\kappa|} (\psi_x^* - 2i\zeta \psi^*)$$

$$Q = -\sqrt{|\kappa|} (\psi_x + 2i\zeta \psi)$$

$$R = 2i\zeta^2 - i|\kappa| \psi^* \psi$$

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \left(-\frac{\phi^2}{2} + \frac{\phi^4}{4} \right)$$

$$\partial_\mu (\partial_\mu \phi) - \frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu^2 \phi + (-\phi + \phi^3) = 0$$

$$\partial_t^2 \phi = \partial_x^2 \phi + (\phi - \phi^3)$$

$$\mathcal{H} = \pi \dot{\phi} - \mathcal{L} \quad ; \quad \pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}$$

$$= \frac{1}{2} (\dot{\phi}^2 + \phi'^2) + \left(\frac{1}{4} (\phi^2 - 2\phi^2) \right)$$

$$= \frac{1}{2} (\dot{\phi}^2 + \phi'^2) + \frac{1}{4} (\phi^2 - 1)^2 \left(-\frac{1}{4} \right)$$

$$H(i,j) = \frac{1}{2} \left(\frac{\phi(i+1,j) - \phi(i,j)}{\Delta t} \right)^2 + \frac{1}{2} \left(\frac{\phi(i,j+1) - \phi(i,j)}{\Delta x} \right)^2 + \frac{1}{4} (\phi^2 - 1)^2$$